Random geometry in the path integral approach to quantum gravity

Three facets of Gravity, Humboldt University of Berlin
Exercises by T. Budd for tutorial on 10 May 2023

Exercise 1.1: Generating functions of maps with boundaries

Recall that the partition function of bipartite planar maps with face weights $q = (q_1, q_2, \ldots)$ is defined as

$$F_0(t,q) = \sum_{\text{maps } \mathfrak{m}} \frac{w(\mathfrak{m})}{|\operatorname{Aut}(\mathfrak{m})|} = \sum_{\text{rooted maps } \mathfrak{m}} \frac{w(\mathfrak{m})}{2|E(\mathfrak{m})|}, \qquad w(\mathfrak{m}) = t^{|V(\mathfrak{m})|} \prod_{f\in F(\mathfrak{m})} q_{\deg f/2}. \tag{1.1}$$

a) Demonstrate that the disk function

$$W^{(\ell)}(t,q) = \sum_{\substack{\text{rooted map } \mathfrak{m}\\ \text{with } \deg(f_r)=2\ell}} w'(\mathfrak{m}),$$

where $w'(\mathfrak{m})$ is the same as $w(\mathfrak{m})$ except that the root face is omitted in the product, is obtained from F_0 via

$$W^{(\ell)}(t,q) = 2\ell \frac{\partial F_0}{\partial q_\ell}. \tag{1.2}$$

Similarly, the generating functions $W^{(\ell_1,\ldots,\ell_n)}(t,q)$ of bipartite planar maps with n distinguished faces ("boundaries") of degrees $2\ell_1,\ldots,2\ell_n$, each with a marked oriented edge in their contour, are obtained by taking further derivatives

$$W^{(\ell_1,\ldots,\ell_n)}(t,q) = 2\ell_n \frac{\partial W^{(\ell_1,\ldots,\ell_{n-1})}}{\partial q_{\ell_n}}. \tag{1.3}$$

In the lecture we saw that $F_0(t,q)$ could be rather explicitly expressed as

$$F_0(t,q) = \frac{1}{2}\int_0^R \frac{\mathrm{d}r}{r}\left[(g_q(r)-t)^2 - (r-t)^2 \mathbb{1}_{\{r<t\}}\right] \tag{1.4}$$

where

$$g_q(r) = r - \sum_{k=1}^{\infty} q_k \frac{1}{2}\binom{2k}{k} r^k, \tag{1.5}$$

and $R(t,q) = \frac{t}{1-q_1} + O(t^2)$ is the power series solution to $g_q(R) = t$.

b) Show that the generating functions for the disk ($n=1$), the cylinder ($n=2$), and the pair of pants ($n=3$) are given by the expressions

$$W^{(\ell_1)}(t,q) = \binom{2\ell_1}{\ell_1}\int_0^R \mathrm{d}r\, g_q'(r)\, r^{\ell_1},$$

$$W^{(\ell_1,\ell_2)}(t,q) = \frac{\ell_1\ell_2}{\ell_1+\ell_2}\binom{2\ell_1}{\ell_1}\binom{2\ell_2}{\ell_2} R^{\ell_1+\ell_2},$$

$$W^{(\ell_1,\ell_2,\ell_3)}(t,q) = \ell_1\ell_2\ell_3\binom{2\ell_1}{\ell_1}\binom{2\ell_2}{\ell_2}\binom{2\ell_3}{\ell_3} R^{\ell_1+\ell_2+\ell_3-1}\frac{\partial R}{\partial t}.$$

Hint: Use that $g_q(R) = t$ repeatedly!

Solution 1.1:

a) Performing the partial derivative explicitly, we find

$$\begin{aligned}
2\ell\frac{\partial F_0}{\partial q_\ell}(t,q) &= \sum_{\text{rooted maps } \mathfrak{m}} \frac{1}{2|E(\mathfrak{m})|} 2\ell \frac{\partial w(\mathfrak{m})}{\partial q_\ell} \\
&= \sum_{\text{rooted maps } \mathfrak{m}} \frac{1}{2|E(\mathfrak{m})|} \sum_{\substack{\text{oriented edges } e \text{ of } \mathfrak{m} \\ \text{belonging to face } f' \text{ of degree } 2\ell}} t^{|V(\mathfrak{m})|} \prod_{f\in F(\mathfrak{m})\setminus\{f'\}} q_{\deg f/2} \\
&= \sum_{\substack{\text{rooted maps } \mathfrak{m} \\ \text{with } \deg(f_\mathrm{r})=2\ell}} \frac{1}{2|E(\mathfrak{m})|} \sum_{\text{oriented edges } e \text{ of } \mathfrak{m}} t^{|V(\mathfrak{m})|} \prod_{f\in F(\mathfrak{m})\setminus\{f_\mathrm{r}\}} q_{\deg f/2} \\
&= \sum_{\substack{\text{rooted maps } \mathfrak{m} \\ \text{with } \deg(f_\mathrm{r})=2\ell}} w'(\mathfrak{m}) = W^{(\ell)}(t,q).
\end{aligned}$$

Here in the third equality we use that the oriented edge e plays the same role as the root edge, except for the constraint that $\deg f = 2\ell$, so we can interchange their roles in the summation.

b) The basic idea is that the partial derivatives on the integration bound R vanish in most situations, because $g_q(R) - t = 0$. For the disk function we furthermore make use of an integration by parts,

$$\begin{aligned}
W^{(\ell_1)}(t,q) = 2\ell_1 \frac{\partial F_0}{\partial q_{\ell_1}} &= \int_0^R \mathrm{d}r (t - g_q(r)) \binom{2\ell_1}{\ell_1} \ell_1 r^{\ell_1 - 1} \\
&= \binom{2\ell_1}{\ell_1} \int_0^R \mathrm{d}r\, g_q'(r)\, r^{\ell_1}.
\end{aligned}$$

For the cylinder, it is best to start from the penultimate expression of the disk function and take a derivative of the factor $t - g_q(r)$ in the integrand,

$$\begin{aligned}
W^{(\ell_1,\ell_2)}(t,q) = 2\ell_2 \frac{\partial W^{(\ell_1)}}{\partial q_{\ell_2}} &= \ell_1\ell_2 \int_0^R \mathrm{d}r \binom{2\ell_1}{\ell_1}\binom{2\ell_2}{\ell_2} r^{\ell_1+\ell_2-1} \\
&= \frac{\ell_1\ell_2}{\ell_1+\ell_2} \binom{2\ell_1}{\ell_1}\binom{2\ell_2}{\ell_2} R^{\ell_1+\ell_2}.
\end{aligned}$$

For the pair of pants we first observe that

$$0 = \frac{\partial}{\partial q_k} g_q(R) = \frac{\partial g_q}{\partial q_k}(R) + g_q'(R)\frac{\partial R}{\partial q_k} \quad \text{and} \quad 1 = \frac{\partial}{\partial t} g_q(R) = g_q'(R)\frac{\partial R}{\partial t},$$

which together imply that

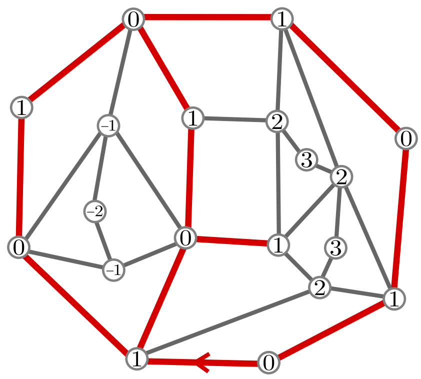
$$\frac{\partial R}{\partial q_k} = -\frac{\partial g_q}{\partial q_k}(R)\frac{\partial R}{\partial t} = \frac{1}{2}\binom{2k}{k} R^k \frac{\partial R}{\partial t}.$$

Then we find

$$\begin{aligned}
W^{(\ell_1,\ell_2,\ell_3)}(t,q) = 2\ell_3 \frac{\partial W^{(\ell_1,\ell_2)}}{\partial q_{\ell_3}} &= 2\ell_1\ell_2\ell_3 \binom{2\ell_1}{\ell_1}\binom{2\ell_2}{\ell_2} R^{\ell_1+\ell_2-1} \frac{\partial R}{\partial q_{\ell_3}} \\
&= \ell_1\ell_2\ell_3 \binom{2\ell_1}{\ell_1}\binom{2\ell_2}{\ell_2}\binom{2\ell_3}{\ell_3} R^{\ell_1+\ell_2+\ell_3-1} \frac{\partial R}{\partial t}.
\end{aligned}$$

Exercise 1.2: Non-generic critical maps: an example

Non-generic critical maps naturally appear when considering two-dimensional quantum gravity coupled to matter. In this exercise we will look at a specific example, that can be understood as a discretization of two-dimensional quantum gravity coupled to a massless scalar field (matter central charge $c = 1$). According to general wisdom this should be in the universality class of $\mathrm{LQG}_{\gamma=2}$ and therefore feature non-generic critical phenomena with index $\alpha = 3/2$.



The discrete model is based on quadrangulations $\mathfrak{q}$ with perimeter 2ℓ, i.e. planar maps with all faces of degree 4 except the root face which has degree 2ℓ. A (colourful) labeling $V(\mathfrak{q}) \to \mathbb{Z}$ of $\mathfrak{q}$ is an assignment of integer labels to the vertices such that:

- the labels at the endpoints of each edge differ by 1;
- each face (except the root face) is incident to three different labels;
- the root face is labeled (alternatingly) by 0 and 1 such that the root edge points from label 0 to 1.

We denote the generating function of labeled quadrangulations by

$$Q^{(\ell)}(v) = \sum_{\substack{\text{labeled} \\ \text{quadrangulations } \mathfrak{q} \\ \text{of perimeter } 2\ell}} v^{|V(\mathfrak{q})|}. \tag{2.1}$$

By convention $Q^{(0)}(v) = v$, counting the map consisting of a single vertex.

a) Keeping only the edges of the labelled quadrangulation $\mathfrak{q}$ that have labels 0 and 1 and dropping all edges that are not in the connected component of the root edge, we obtain the *gasket* $g(\mathfrak{q})$ of $\mathfrak{q}$. Argue that for any (rooted bipartite) map $\mathfrak{m}$ of perimeter 2ℓ, we have

$$\sum_{\substack{\text{labeled} \\ \text{quadrangulations } \mathfrak{q} \\ \text{of perimeter } 2\ell \\ g(\mathfrak{q})=\mathfrak{m}}} v^{|V(\mathfrak{q})|} = \prod_{f\in F(\mathfrak{m})\setminus\{f_r\}} q_{\deg f/2}(v), \qquad q_k(v) := 2\sum_{p=0}^{\infty}\binom{p+k-1}{p} Q^{(p)}(v). \tag{2.2}$$

Hence, $Q^{(\ell)}(v)$ is related to the disk function $W^{(\ell)}(t,q)$ of Boltzmann maps via $Q^{(\ell)}(v) = W^{(\ell)}(v, q(v))$.

b) Show that the corresponding potential $V'(x) = x - \sum_{k=1}^{\infty} q_k x^{2k-1}$ and disk function $W(x) = \sum_{\ell=0}^{\infty} W^{(\ell)} x^{-2\ell-1}$ are related (for x in a sufficiently small neighbourhood of $1/\sqrt{2}$) via

$$1 - \frac{V'(x)}{x} = 2\frac{W(\sqrt{1-x^2})}{\sqrt{1-x^2}}. \tag{2.3}$$

Therefore $(W(x) - \frac{1}{2}V'(x))/x$ is symmetric under $x \to \sqrt{1-x^2}$.

By the one-cut assumption there exists a power series $M(x)$ and $R(v)$ such that

$$\frac{2}{x}(W(x) - \frac{1}{2}V'(x)) = M(x)\sqrt{1 - \frac{4R(v)}{x^2}}. \tag{2.4}$$

A natural guess is that

$$M(x) = \sqrt{1 - \frac{4R}{1-x^2}} \tag{2.5}$$

and this is confirmed by the explicit formula for $Q^{(\ell)}(v)$ obtained in [Bousquet-Mélou, Elvey Price, arXiv:1803.08265].

c) More challenging: Criticality of the gasket requires $M(x)$ to vanish at the branch cut $x = \pm 2\sqrt{R}$, so $R = 1/8$. Show that in this case $W^{(\ell)} \sim \frac{1}{4\pi}2^{-\ell}\ell^{-2}$ as $\ell \to \infty$, and therefore the gasket is non-generic critical with exponent $\alpha = 3/2$.

Solution 1.2:

a) It should be clear that the gasket of $\mathfrak{q}$ is a rooted bipartite map, and that every bipartite rooted map can be obtained in this fashion (put a single vertex with label 2 in every face and connect it to the vertices with label 1.) Let's consider a single (non-root) face of the gasket of degree $2k$. There are exactly two options for the k quadrangles of $\mathfrak{q}$ that are adjacent to an edge of the gasket: either they all have label $0, 1, 0, -1$ or they have label $1, 0, 1, 2$. Let's focus on the latter case. There can be p further quadrangles with labels $1, 2, 3, 2$ that touch the gasket in a vertex of label 1. There are precisely $\binom{\ell+k-1}{\ell}$ ways to distribute these in between the k gaps. The sides with label 2 and 3 border a labeled quadrangulation of perimeter $2p$, which after shifting the labels by -2 is of the same type as $\mathfrak{q}$. Hence the effective weight

$$\sum_{p=0}^{\infty} 2\binom{p+k-1}{p} Q^{(p)}(v)$$

per face of degree $2k$ in the gasket.

b)

$$\begin{aligned}
1 - \frac{V'(x)}{x} &= \sum_{k=1}^{\infty} x^{2k-2} q_k \\
&= 2\sum_{p=0}^{\infty} Q^{(p)}(v) \sum_{k=1}^{\infty} x^{2k-2}\binom{p+k-1}{p} \\
&= 2\sum_{p=0}^{\infty} Q^{(p)}(v)(1-x^2)^{-p-1} \\
&= 2\frac{W(\sqrt{1-x^2})}{\sqrt{1-x^2}}.
\end{aligned}$$

So that

$$\frac{W(\sqrt{1-x^2}) - \frac{1}{2}V'(\sqrt{1-x^2})}{\sqrt{1-x^2}} = \frac{W(x) - \frac{1}{2}V'(x)}{x}$$

c) $M(2\sqrt{R}) = \sqrt{1 - 4R/(1-4R)} = 0$ when $R = 1/8$. We should be careful that the left-hand side of (2.4) is convergent only on the circle $|x| = 2\sqrt{R} = 1/\sqrt{2}$, but that is enough to extract $W^{(\ell)}$ via contour integration,

$$\begin{aligned}
W^{(\ell)} &= \frac{1}{2\pi i}\int_{|x|=2\sqrt{R}} x^{2\ell} W(x)\mathrm{d}x \\
&= \frac{1}{2\pi i}\int_{|x|=2\sqrt{R}} x^{2\ell}\frac{x}{2}\sqrt{1-\frac{1/2}{1-x^2}}\sqrt{1-\frac{1/2}{x^2}}\mathrm{d}x \\
&= \frac{2^{-\ell}}{8\pi}\int_0^{2\pi} e^{i(2\ell+2)\phi}\sqrt{1-\frac{1/2}{1-\frac{1}{2}e^{2i\phi}}}\sqrt{1-e^{-2i\phi}}\mathrm{d}\phi \\
&= \frac{2^{-\ell}}{4\pi}\int_0^{2\pi} e^{i(2\ell+2)\phi}\frac{|\sin\phi|}{\sqrt{2-e^{2i\phi}}}\mathrm{d}\phi
\end{aligned}$$

The dominant contribution for large ℓ comes from the $|\sin\phi|/\sqrt{2-e^{2i\phi}} = |\phi| + O(\phi^2)$, which leads to

$$W^{(\ell)} \sim \frac{1}{4\pi}2^{-\ell}\ell^{-2}.$$