Random geometry in the path integral approach to quantum gravity

Three facets of Gravity, Humboldt University of Berlin
Exercises by T. Budd for tutorial on 10 May 2023

Exercise 1.1: Generating functions of maps with boundaries

Recall that the partition function of bipartite planar maps with face weights $q = (q_1, q_2, \ldots)$ is defined as

$$F_0(t, q) = \sum_{\text{maps } \mathfrak{m}} \frac{w(\mathfrak{m})}{|\operatorname{Aut}(\mathfrak{m})|} = \sum_{\text{rooted maps } \mathfrak{m}} \frac{w(\mathfrak{m})}{2|E(\mathfrak{m})|}, \qquad w(\mathfrak{m}) = t^{|V(\mathfrak{m})|} \prod_{f \in F(\mathfrak{m})} q_{\deg f/2}. \tag{1.1}$$

a) Demonstrate that the disk function

$$W^{(\ell)}(t, q) = \sum_{\substack{\text{rooted map } \mathfrak{m} \\ \text{with } \deg(f_r)=2\ell}} w'(\mathfrak{m}),$$

where $w'(\mathfrak{m})$ is the same as $w(\mathfrak{m})$ except that the root face is omitted in the product, is obtained from F_0 via

$$W^{(\ell)}(t, q) = 2\ell \frac{\partial F_0}{\partial q_\ell}. \tag{1.2}$$

Similarly, the generating functions $W^{(\ell_1,\ldots,\ell_n)}(t, q)$ of bipartite planar maps with n distinguished faces ("boundaries") of degrees $2\ell_1, \ldots, 2\ell_n$, each with a marked oriented edge in their contour, are obtained by taking further derivatives

$$W^{(\ell_1,\ldots,\ell_n)}(t, q) = 2\ell_n \frac{\partial W^{(\ell_1,\ldots,\ell_{n-1})}}{\partial q_{\ell_n}}. \tag{1.3}$$

In the lecture we saw that $F_0(t, q)$ could be rather explicitly expressed as

$$F_0(t, q) = \frac{1}{2} \int_0^R \frac{\mathrm{d}r}{r} \left[(g_q(r) - t)^2 - (r - t)^2 \mathbb{1}_{\{r<t\}} \right] \tag{1.4}$$

where

$$g_q(r) = r - \sum_{k=1}^{\infty} q_k \frac{1}{2} \binom{2k}{k} r^k, \tag{1.5}$$

and $R(t, q) = \frac{t}{1-q_1} + O(t^2)$ is the power series solution to $g_q(R) = t$.

b) Show that the generating functions for the disk ($n = 1$), the cylinder ($n = 2$), and the pair of pants ($n = 3$) are given by the expressions

$$W^{(\ell_1)}(t, q) = \binom{2\ell_1}{\ell_1} \int_0^R \mathrm{d}r\, g_q'(r)\, r^{\ell_1},$$
$$W^{(\ell_1,\ell_2)}(t, q) = \frac{\ell_1 \ell_2}{\ell_1 + \ell_2} \binom{2\ell_1}{\ell_1} \binom{2\ell_2}{\ell_2} R^{\ell_1+\ell_2},$$
$$W^{(\ell_1,\ell_2,\ell_3)}(t, q) = \ell_1 \ell_2 \ell_3 \binom{2\ell_1}{\ell_1} \binom{2\ell_2}{\ell_2} \binom{2\ell_3}{\ell_3} R^{\ell_1+\ell_2+\ell_3-1} \frac{\partial R}{\partial t}.$$

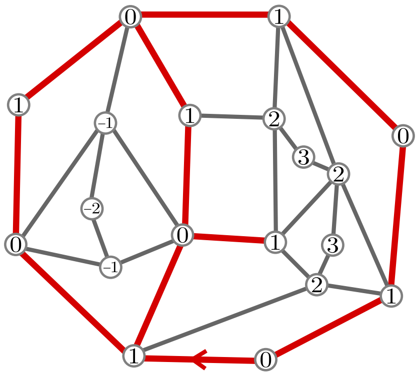
Hint: Use that $g_q(R) = t$ repeatedly!

Exercise 1.2: Non-generic critical maps: an example

Non-generic critical maps naturally appear when considering two-dimensional quantum gravity coupled to matter. In this exercise we will look at a specific example, that can be understood as a discretization of two-dimensional quantum gravity coupled to a massless scalar field (matter central charge $c = 1$). According to general wisdom this should be in the universality class of $\mathrm{LQG}_{\gamma=2}$ and therefore feature non-generic critical phenomena with index $\alpha = 3/2$.

The discrete model is based on quadrangulations $\mathfrak{q}$ with perimeter 2ℓ, i.e. planar maps with all faces of degree 4 except the root face which has degree 2ℓ. A (colourful) labeling $V(\mathfrak{q}) \to \mathbb{Z}$ of $\mathfrak{q}$ is an assignment of integer labels to the vertices such that:

- the labels at the endpoints of each edge differ by 1;
- each face (except the root face) is incident to three different labels;
- the root face is labeled (alternatingly) by 0 and 1 such that the root edge points from label 0 to 1.



We denote the generating function of labeled quadrangulations by

$$Q^{(\ell)}(v) = \sum_{\substack{\text{labeled} \\ \text{quadrangulations } \mathfrak{q} \\ \text{of perimeter } 2\ell}} v^{|V(\mathfrak{q})|}. \tag{2.1}$$

By convention $Q^{(0)}(v) = v$, counting the map consisting of a single vertex.

a) Keeping only the edges of the labelled quadrangulation $\mathfrak{q}$ that have labels 0 and 1 and dropping all edges that are not in the connected component of the root edge, we obtain the *gasket* $g(\mathfrak{q})$ of $\mathfrak{q}$. Argue that for any (rooted bipartite) map $\mathfrak{m}$ of perimeter 2ℓ, we have

$$\sum_{\substack{\text{labeled} \\ \text{quadrangulations } \mathfrak{q} \\ \text{of perimeter } 2\ell \\ g(\mathfrak{q})=\mathfrak{m}}} v^{|V(\mathfrak{q})|} = \prod_{f \in F(\mathfrak{m}) \setminus \{f_r\}} q_{\deg f/2}(v), \qquad q_k(v) := 2\sum_{p=0}^{\infty} \binom{p+k-1}{p} Q^{(p)}(v). \tag{2.2}$$

Hence, $Q^{(\ell)}(v)$ is related to the disk function $W^{(\ell)}(t,q)$ of Boltzmann maps via $Q^{(\ell)}(v) = W^{(\ell)}(v, q(v))$.

b) Show that the corresponding potential $V'(x) = x - \sum_{k=1}^{\infty} q_k x^{2k-1}$ and disk function $W(x) = \sum_{\ell=0}^{\infty} W^{(\ell)} x^{-2\ell-1}$ are related (for x in a sufficiently small neighbourhood of $1/\sqrt{2}$) via

$$1 - \frac{V'(x)}{x} = 2\frac{W(\sqrt{1-x^2})}{\sqrt{1-x^2}}. \tag{2.3}$$

Therefore $(W(x) - \frac{1}{2}V'(x))/x$ is symmetric under $x \to \sqrt{1-x^2}$.

By the one-cut assumption there exists a power series $M(x)$ and $R(v)$ such that

$$\frac{2}{x}(W(x) - \frac{1}{2}V'(x)) = M(x)\sqrt{1 - \frac{4R(v)}{x^2}}. \tag{2.4}$$

A natural guess is that

$$M(x) = \sqrt{1 - \frac{4R}{1-x^2}} \tag{2.5}$$

and this is confirmed by the explicit formula for $Q^{(\ell)}(v)$ obtained in [Bousquet-Mélou, Elvey Price, arXiv:1803.08265].

c) More challenging: Criticality of the gasket requires $M(x)$ to vanish at the branch cut $x = \pm 2\sqrt{R}$, so $R = 1/8$. Show that in this case $W^{(\ell)} \sim \frac{1}{4\pi} 2^{-\ell} \ell^{-2}$ as $\ell \to \infty$, and therefore the gasket is non-generic critical with exponent $\alpha = 3/2$.