Corner Structure of Four-Dimensional General Relativity in the Coframe Formalism

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Definition

An n-graded symplectic manifold is a pair (M, ϖ) where M is a graded manifold and ϖ is a closed nondegenerate two-form on M of homogenous degree n and parity n mod 2.

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A differential graded manifold (shortly, a dg manifold) is a pair (M, Q) such that Q is a cohomological vector field on a graded manifold M, i.e. an odd vector field Q of degree $+1$ satisfying $[Q, Q] = 0$. (Note that Q defines a differential on $C^\infty(M)$.)

BF^{m}V theory

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Definition

A dg manifold with a compatiple symplectic structure, i.e., with $L_Q \varpi = 0$, is called a differential graded symplectic manifold (shortly, a dg symplectic manifold).

Definition

We will always assume that Q is hamiltonian, namely, that there is an $S \in C^\infty(M)_{\text{hamiltonian}}$ such that $\iota_Q \varpi = \mathrm{d}S$ and $\{S, S\} = 0$ (the master equation). If ϖ has degree n, then S has degree $m = n + 1$. In this case, we call the triple (M, ϖ, S) a BF^{m}V manifold.

P_∞ structures from the BF^{2}V formalism

In the case of a BF^{2}V manifold, ϖ is an odd symplectic form of degree $+1$. We start with the finite-dimensional case.

Polarization

(M, ϖ) is always symplectomorphic to a shifted cotangent bundle $T^*[1]N$, with canonical symplectic structure, for some graded manifold N . We call this choice of N a polarization.

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P_∞ structure

The Poisson algebra of functions on $T^*[1]N$ can be canonically identified with the algebra of multivector fields on N with the Schouten bracket. The function S, of degree $+2$, then corresponds to a linear combination $\pi = \pi_0 + \pi_1 + \pi_2 + \cdots$, where π_i is an i-vector field of degree $2 - i$ on N. The master equation $\{S, S\} = 0$ corresponds to the equations

$$
\begin{aligned}
[\pi_0, \pi_1] &= 0, \\
[\pi_0, \pi_2] + \frac{1}{2}[\pi_1, \pi_1] &= 0, \\
[\pi_0, \pi_3] + [\pi_1, \pi_2] &= 0, \\
[\pi_0, \pi_4] + [\pi_1, \pi_3] + \frac{1}{2}[\pi_2, \pi_2] &= 0, \\
&\cdots
\end{aligned}
$$

π is called a P_∞ structure on N (this stands for Poisson structure up to coherent homotopies). This structure is called curved if $\pi_0 \neq 0$.

Multibrackets

L_∞-algebra

The π_is, applied to the differentials of i functions on N, define multibrackets $\{\ \}_i$ on $C^\infty(N)$ which in turn define a (curved) L_∞-algebra. Moreover, they are graded derivations w.r.t. each argument. The multibrackets may also be defined as derived brackets

$$\{f_1, \dots, f_i\}_i = [[[[\cdots[\pi_i, f_1], f_2], \dots], f_i] = P[[[[[\cdots[\pi, f_1], f_2], \dots], f_i],$$

where P is the projection from multivector fields to functions. In particular, we have

$$\{\}_0 = \pi_0, \qquad \{f\}_1 = \pi_1(f), \qquad \{f, g\}_2 = [[\pi_2, f], g].$$

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Generalizations

The above structure may be generalized as follows.

Weak polarization

Suppose we have a splitting $C^\infty(M) = \mathfrak{p} \oplus \mathfrak{h}$ into Poisson subalgebras with $\mathfrak{h}$ abelian (i.e., $\mathfrak{p} \cdot \mathfrak{p} \subseteq \mathfrak{p}$, $\mathfrak{h} \cdot \mathfrak{h} \subseteq \mathfrak{h}$, $\{\mathfrak{p}, \mathfrak{p}\} \subseteq \mathfrak{p}$, $\{\mathfrak{h}, \mathfrak{h}\} = 0$). Let P be the projection $C^\infty(M) \to \mathfrak{h}$. Then the multibrackets

$$\{f_1, \ldots, f_i\}_i := P\{\cdots\{S, f_1\}, f_2\}, \ldots\}, f_i\}$$

make $\mathfrak{h}$ into a P_∞-algebra. We call the more general choice of $(\mathfrak{p}, \mathfrak{h})$ a weak polarization.

ϖ degenerate

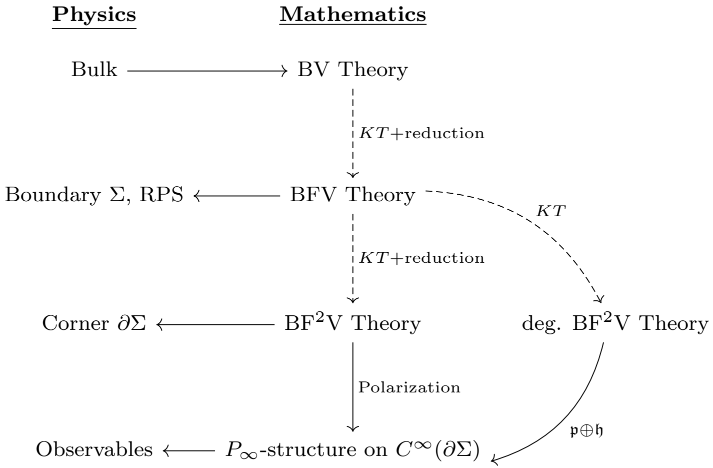
In this case we consider a splitting, with the above properties, of the -1-Poisson algebra of hamiltonian functions: $C^\infty_{\text{hamiltonian}}(M) = \mathfrak{p} \oplus \mathfrak{h}$.

Infinite-dimensional case

- M is symplectomorphic to a symplectic subbundle of $T^*[1]N$, for some infinite-dimensional graded manifold N.
- Not every function is hamiltonian. We can anyway define the derived brackets, as before, on $C^\infty_{\text{hamiltonian}}(N) := C^\infty(N) \cap C^\infty_{\text{hamiltonian}}(M)$.
- The algebraic version for weak polarizations and its extension to the degenerate case works verbatim as before.

Summary

Input: Gauge field theory on manifold with corners.



Yang–Mills theory

BFV theory

- $\varpi^{\partial} = \int_{\Sigma}(\delta B\, \delta A + \delta b\, \delta c)$
- $S^{\partial} = \int_{\Sigma} \left(c\, \mathrm{d}_A B + \frac{1}{2} b[c, c]\right)$

BF^{2}V theory

- $\varpi^{\partial\partial} = \int_{\partial\Sigma} \delta c\, \delta B$
- $S^{\partial\partial} = \int_{\Sigma} \frac{1}{2} B[c, c]$.

Poisson Structures

1. If we regard $\mathcal{F}_{\partial\Sigma}$ as $T^*[1](\Omega^2(\partial\Sigma) \otimes \mathfrak{g})$, we then interpret $S^{\partial\partial}$ as the Poisson bivector field
$$\pi_2 = \int_{\Sigma} \frac{1}{2} B \left[\frac{\delta}{\delta B}, \frac{\delta}{\delta B}\right].$$
2. The other natural polarization consists in realizing $\mathcal{F}_{\partial\Sigma}$ as $T^*[1](C^{\infty}(\partial\Sigma)[1] \otimes \mathfrak{g})$. In this case we interpret $S^{\partial\partial}$ as the cohomological vector field
$$\pi_1 = \int_{\Sigma} \frac{1}{2} [c, c] \frac{\delta}{\delta c},$$
which gives $C^{\infty}(\partial\Sigma)[1] \otimes \mathfrak{g}$ the structure of a P_{∞}-manifold.

BF theory

In BF theory in 4 dimensions there are two classical fields: a $\mathfrak{g}$-connection A and a $\mathfrak{g}$-valued 2-form B. Here $\mathfrak{g}$ is, a Lie algebra endowed with a nondegenerate, invariant inner product

BFV theory

- $\varpi^{\partial} = \int_{\Sigma}(\delta A^{+}\,\delta c + \delta B\,\delta A + \delta\tau\,\delta B^{+} + \delta\phi\,\delta\tau^{+})$
- $S^{\partial} = \int_{\Sigma}\left(\frac{1}{2}A^{+}[c,c] + B\,\mathrm{d}_A c + \tau\,(F_A + [c,B^{+}]) + \phi\,(\mathrm{d}_A B^{+} + [c,\tau^{+}]) + \Lambda\,(B\tau + A^{+}\phi)\right)$

If Σ has a boundary, we get a BF^{2}V theory on $\partial\Sigma$

BF^{2}V theory

- $\varpi^{\partial\partial} = \int_{\partial\Sigma}(\delta B\,\delta c + \delta\tau\,\delta A + \delta\phi\,\delta B^{+})$.
-

$$
\begin{aligned}
S^{\partial\partial} &= \int_{\partial\Sigma}\left(\frac{1}{2}B[c,c] + \tau\,\mathrm{d}_A c + \phi\,(F_A + [c,B^{+}]) + \Lambda\left(\frac{1}{2}\tau\tau + B\phi\right)\right) \\
&= \int_{\partial\Sigma}\left(\frac{1}{2}B[c,c] + \tau\,(\mathrm{d}_{A_0}c + [a,c]) + \phi\,\left(F_{A_0} + \mathrm{d}_{A_0}a + \frac{1}{2}[a,a] + [c,B^{+}]\right)\right) \\
&\qquad + \Lambda\left(\frac{1}{2}\tau\tau + B\phi\right)
\end{aligned}
$$

where A_0 is a reference connection and $a = A - A_0$.

Poisson structures

1. Lagrangian submanifold: $\{c = \phi = \tau = 0\}$; This corresponds to having $\pi = \pi_1 + \pi_2$ with

$$\pi_1 = \int_{\partial\Sigma} (F_A + \Lambda B) \frac{\delta}{\delta B^+},$$
$$\pi_2 = \int_{\partial\Sigma} \left(\frac{1}{2} B \left[\frac{\delta}{\delta B}, \frac{\delta}{\delta B} \right] + \frac{\delta}{\delta a} \mathrm{d}_{A_0} \frac{\delta}{\delta B} + a \left[\frac{\delta}{\delta a}, \frac{\delta}{\delta B} \right] + B^+ \left[\frac{\delta}{\delta B^+}, \frac{\delta}{\delta B} \right] + \frac{1}{2} \Lambda \frac{\delta}{\delta a} \frac{\delta}{\delta a} \right).$$

   In other words, we split functions on $\mathcal{F}_{\partial\Sigma}$ as $\mathfrak{p} \oplus \mathfrak{h}$ with $\mathfrak{p}$ the subalgebra of functions of positive degree and $\mathfrak{h}$ the subalgebra of functions of nonpositive degree, and the construction turns $\mathfrak{h}$ into a differential graded Poisson algebra. The degree zero part $\mathfrak{h}_0$, consisting of functions on $\mathcal{A}(\partial\Sigma) \oplus \Omega^2(\partial\Sigma) \otimes \mathfrak{g} \ni (A, B)$, is a Poisson subalgebra.

2. Lagrangian submanifold $\{c = B^\dagger = 0, A = A_0\}$; In this case we have $\pi = \pi_0 + \pi_1 + \pi_2$ with

$$\pi_0 = \int_{\partial\Sigma} \left(\phi F_{A_0} + \Lambda \left(\frac{1}{2} \tau\tau + B\phi \right) \right),$$
$$\pi_1 = \int_{\partial\Sigma} \left(\mathrm{d}_{A_0} \tau \frac{\delta}{\delta B} + \mathrm{d}_{A_0} \phi \frac{\delta}{\delta \tau} \right),$$
$$\pi_2 = \int_{\partial\Sigma} \left(\frac{1}{2} B \left[\frac{\delta}{\delta B}, \frac{\delta}{\delta B} \right] + \tau \left[\frac{\delta}{\delta \tau}, \frac{\delta}{\delta B} \right] + \frac{1}{2} \phi \left[\frac{\delta}{\delta \tau}, \frac{\delta}{\delta \tau} \right] + \phi \left[\frac{\delta}{\delta \phi}, \frac{\delta}{\delta B} \right] \right).$$

   This makes $C^\infty(\widetilde{\mathcal{B}})$ into a curved P_∞ algebra.

BF theory – Poisson structures II

There is a P_∞ subalgebra generated by the following linear local observables:

$$J_\alpha = \int_{\partial\Sigma} \alpha B, \quad M_\beta = \int_{\partial\Sigma} \beta\tau, \quad K_\gamma = \int_{\partial\Sigma} \gamma\phi,$$

where α, β, γ are $\mathfrak{g}$-valued 0-, 1-, and 2−forms, respectively.

Brackets

$$\{\}_0 = \int_{\partial\Sigma} \left(\phi F_{A_0} + \Lambda\left(\frac{1}{2}\tau\tau + B\phi\right)\right)$$

$$\{J_\alpha\}_1 = M_{\mathrm{d}_{A_0}\alpha}, \quad \{M_\beta\}_1 = K_{\mathrm{d}_{A_0}\beta}, \quad \{K_\gamma\}_1 = 0,$$

$$\{J_\alpha, J_{\widetilde{\alpha}}\}_2 = J_{[\alpha,\widetilde{\alpha}]}, \quad \{J_\alpha, M_\beta\}_2 = M_{[\alpha,\beta]}, \quad \{J_\alpha, K_\gamma\}_2 = K_{[\alpha,\gamma]},$$

$$\{M_\beta, M_{\widetilde{\beta}}\}_2 = K_{[\beta,\widetilde{\beta}]}, \quad \{M_\beta, K_\gamma\}_2 = 0, \quad \{K_\gamma, K_{\widetilde{\gamma}}\}_2 = 0.$$

Note that, when $\Lambda = 0$, the above algebra closes also under the nullary operation, since we can write

$$\{\}_0 = K_{F_{A_0}}.$$

Gravity theory (Coframe formulation)

BFV theory

$$S^\partial = \int_\Sigma \Big(ce\mathrm{d}_\omega e + \iota_\xi e e F_\omega + \lambda \epsilon_n e F_\omega + \frac{1}{3!}\lambda\epsilon_n \Lambda e^3 + \frac{1}{2}[c,c]\gamma^\dagger - L_\xi^\omega c\gamma^\dagger + \frac{1}{2}\iota_\xi\iota_\xi F_\omega \gamma^\dagger$$
$$+ [c,\lambda\epsilon_n]y^\dagger - L_\xi^\omega(\lambda\epsilon_n)y^\dagger - \frac{1}{2}\iota_{[\xi,\xi]} e y^\dagger \Big),$$
$$\varpi^\partial = \int_\Sigma \Big(e\delta e \delta\omega + \delta c \delta \gamma^\dagger - \delta\omega\delta(\iota_\xi\gamma^\dagger) + \delta\lambda\epsilon_n\delta y^\dagger + \iota_{\delta\xi}\delta(ey^\dagger) \Big).$$

Proposition

The BFV theory $\mathfrak{F}^{(1)}_{PC} = (\mathcal{F}^\partial_{PC}, S^\partial_{PC}, \varpi^\partial_{PC}, Q^\partial_{PC})$ is not 1-extendable.

BF^{2}V theory

We consider the particular case $\xi^m = 0$, $\lambda = 0$.

- $\varpi^{\partial\partial} = \int_\Gamma \left(\delta[c]\delta E - \iota_{\delta\xi}\delta P\right)$
  where E is a pure tensor and $[c]$ denotes the equivalence class of elements $c \in \Omega^{0,2}_{\partial\partial}[1]$ under the equivalence relation $c + d \sim c$ for $d \in \Omega^{0,2}_{\partial\partial}[1]$ such that $ed = 0$.
- $S^{\partial\partial}_{\omega_0} = \int_\Gamma \left(\frac{1}{2}[[c],[c]]E + \iota_\xi(E)\mathrm{d}_{\omega_0}[c] - \frac{1}{2}\iota_{[\xi,\xi]}P + \frac{1}{2}E\iota_\xi\iota_\xi F_{\omega_0}\right).$

Poisson$_\infty$ structures - I

We choose $\{P = c = 0\}$ as the lagrangian submanifold. From the action we get $\pi = \pi_0 + \pi_1 + \pi_2$ with

$$\pi_0 = \int_\Gamma \frac{1}{2} E \iota_\xi \iota_\xi F_{\omega_0},$$
$$\pi_1 = \int_\Gamma \left(\iota_\xi E \mathrm{d}_{\omega_0} \frac{\delta}{\delta E} - \frac{1}{2} \iota_{[\xi,\xi]} \frac{\delta}{\delta \xi} \right),$$
$$\pi_2 = \int_\Gamma \frac{1}{2} \left[\frac{\delta}{\delta E}, \frac{\delta}{\delta E} \right] E.$$

These equip $C^\infty(\mathcal{N})$ with the structure of a curved Poisson$_\infty$ algebra.
We consider a subalgebra of linear functionals of the form:

$$J_\varphi = \int_\Gamma \varphi E, \qquad M_Y = \int_\Gamma Y \iota_\xi E, \qquad K_Z = \int_\Gamma \frac{1}{2} Z \iota_\xi \iota_\xi E.$$

The derived brackets are as follows

$$\{\}_0 = K_{F_{\omega_0}},$$
$$\{J_\varphi\}_1 = M_{\mathrm{d}_{\omega_0}\varphi}, \qquad \{M_Y\}_1 = K_{\mathrm{d}_{\omega_0} Y}, \qquad \{K_Z\}_1 = 0,$$
$$\{J_\varphi, J_{\varphi'}\}_2 = J_{[\varphi,\varphi']}, \qquad \{J_\varphi, M_Y\}_2 = M_{[\varphi,Y]}, \qquad \{J_\varphi, K_Z\}_2 = K_{[\varphi,Z]},$$
$$\{M_Y, M_{Y'}\}_2 = K_{[Y,Y']}, \qquad \{M_Y, K_Z\}_2 = 0, \qquad \{K_Z, K_{Z'}\}_2 = 0.$$

Poisson$_\infty$ structures - II

We choose $\{\xi = c = 0\}$ as the lagrangian submanifold. From the action we get $\pi = \pi_2$ with

$$\pi_2 = \int_\Gamma \left(\frac{1}{2} \left[\frac{\delta}{\delta E}, \frac{\delta}{\delta E} \right] E + \iota_{\frac{\delta}{\delta P}}(E) \mathrm{d}_{\omega_0} \frac{\delta}{\delta E} - \frac{1}{2} \iota_{[\frac{\delta}{\delta P}, \frac{\delta}{\delta P}]} P + \frac{1}{2} E \iota_{\frac{\delta}{\delta P}} \iota_{\frac{\delta}{\delta P}} F_{\omega_0} \right),$$

which equips $C^\infty(\mathcal{N})$ with the structure of a Poisson algebra.
As before we can consider a subalgebra of linear functionals. Let

$$F_X = \int_\Gamma \iota_X P \quad \text{and} \quad J_\varphi = \int_\Gamma \varphi E.$$

Their binary brackets are as follows:

$$\{J_\varphi, J_{\varphi'}\}_2 = J_{[\varphi, \varphi']}, \quad \{J_\varphi, F_X\}_2 = J_{\iota_X \mathrm{d}_{\omega_0} \varphi}, \quad \{F_X, F_{X'}\}_2 = F_{[X, X']} + J_{\iota_X \iota_{X'} F_{\omega_0}}.$$

Theorem

The BF^2V structure of the tangent theory on a corner Γ induces an Atiyah algebroid structure on $\mathrm{ad}\, P \oplus T\Gamma$.

References

I. A. Batalin and G. A. Vilkovisky. “Gauge algebra and quantization”. *Physics Letters B* 102.1 (June 1981), pp. 27–31. DOI: 10.1016/0370-2693(81)90205-7.

Y. Kosmann-Schwarzbach. “From Poisson algebras to Gerstenhaber algebras”. *Annales de l'institut Fourier*. Vol. 46. 5. 1996, pp. 1243–1274.

T. Voronov. “Higher derived brackets and homotopy algebras”. *Journal of Pure and Applied Algebra* 202.1-3 (Nov. 2005), pp. 133–153. ISSN: 0022-4049. DOI: 10.1016/j.jpaa.2005.01.010.

F. Schätz. “BFV-Complex and Higher Homotopy Structures”. *Communications in Mathematical Physics* 286.2 (Dec. 2008), p. 399. ISSN: 1432-0916.

A. S. Cattaneo, P. Mnev, and N. Reshetikhin. “Classical BV Theories on Manifolds with Boundary”. *Communications in Mathematical Physics* 332.2 (2014), pp. 535–603. ISSN: 1432-0916. DOI: 10.1007/s00220-014-2145-3.

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Thank you!