Slice decomposition of hypermaps

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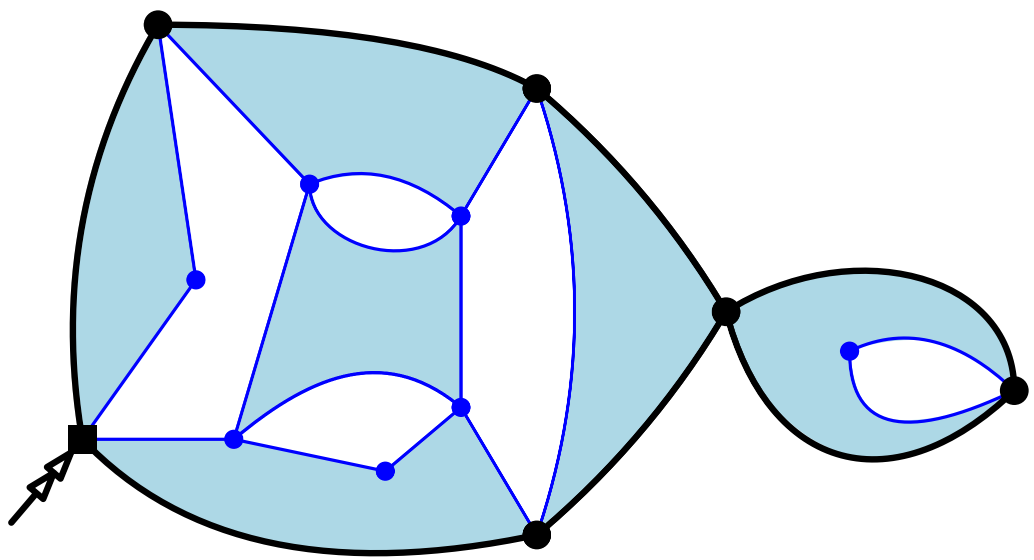
joint work with **Jérémie Bouttier** (CEA, Saclay)

3 facets of gravity, Berlin, May 2023

Hypermaps



An **hypermap** is a planar map in which the faces can be properly bicolored.

Why “hypermap” ?
$\rightarrow$ Extend the notion of hypergraphs to maps.
$\rightarrow$ Blue faces can been as **hyper-edges** which connect several vertices.

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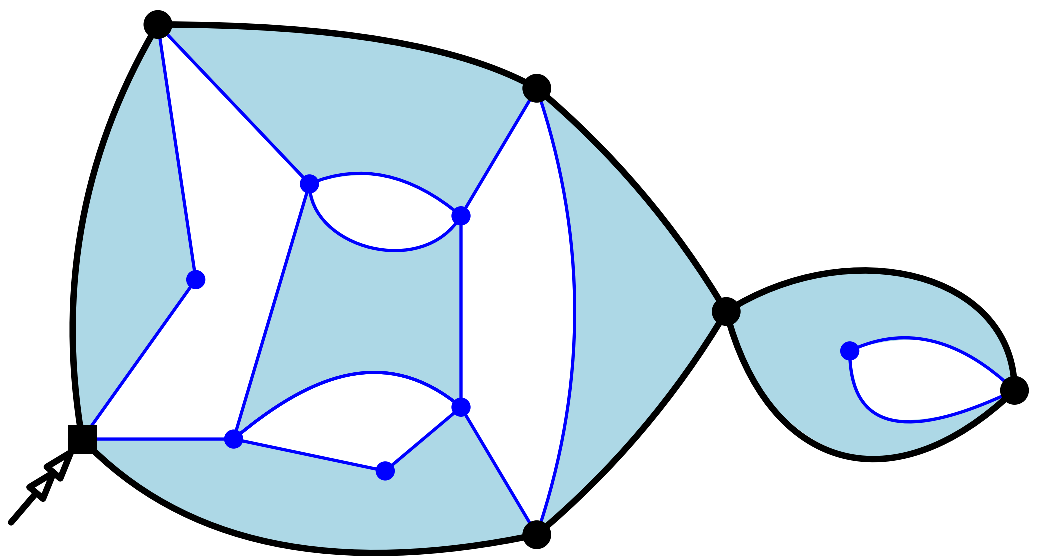

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Hypermaps generalize classical maps:

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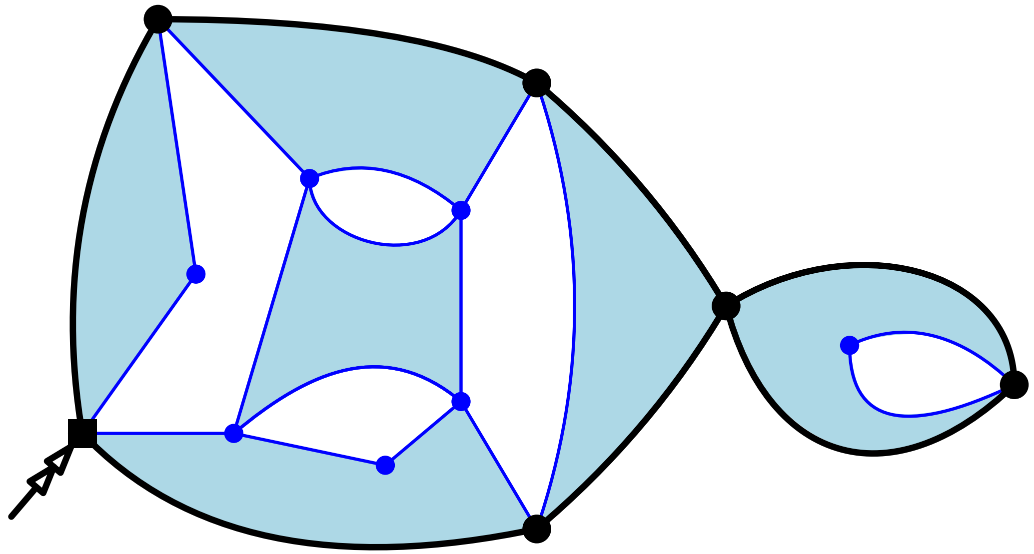

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To each hypermap m, we associate the weight:

$$w(\mathrm{m}) := t^{|\text{vertices of } \mathrm{m}|} \prod_{f \in F^\circ} t^\circ_{\deg(f)} \prod_{f \in F^\bullet} t^\bullet_{\deg(f)}$$

Goal of this talk: enumerate weighted hypermaps **bijectively**.

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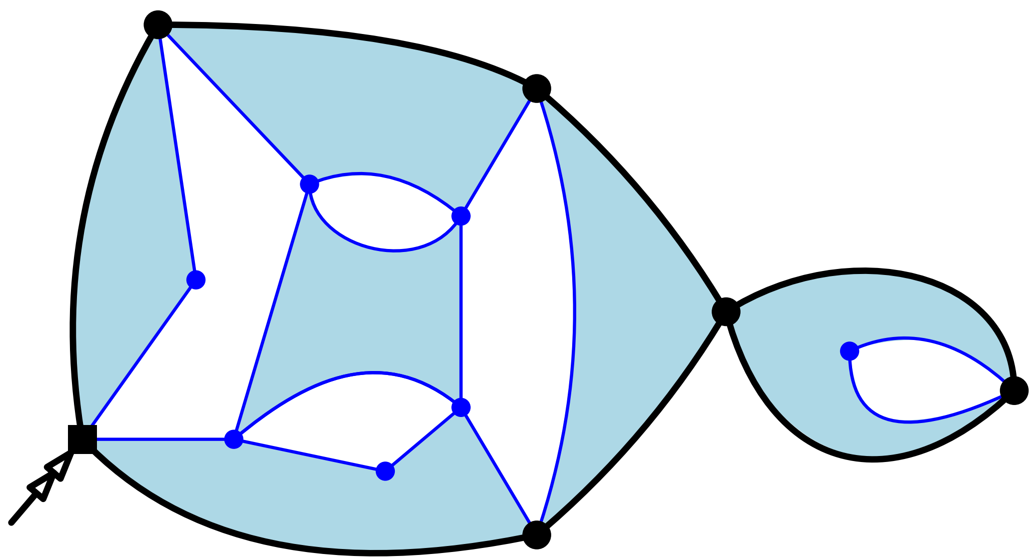

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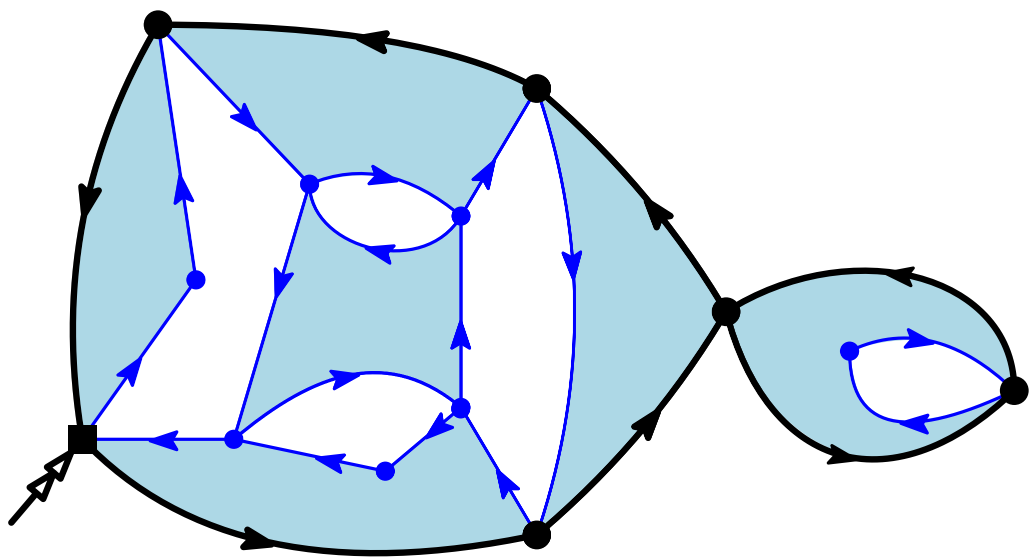

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Motivations and existing litterature

Hypermaps generalize maps, also additional motivations from **theoretical physics**:

- **2-matrix models** ([Itzykson-Zuber 1980], [Eynard et al. 2000's])
- **Ising model on maps** ([Kazakov 1986])
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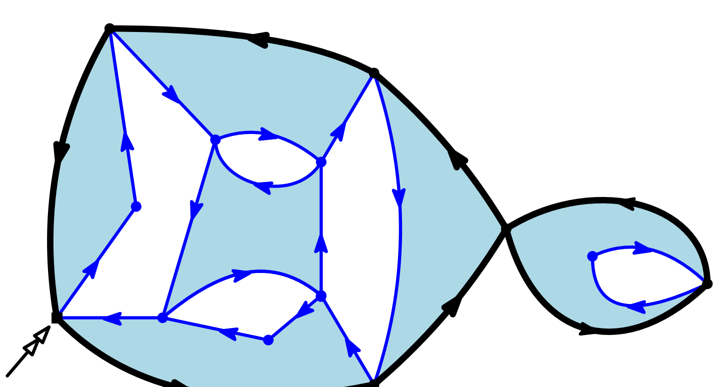
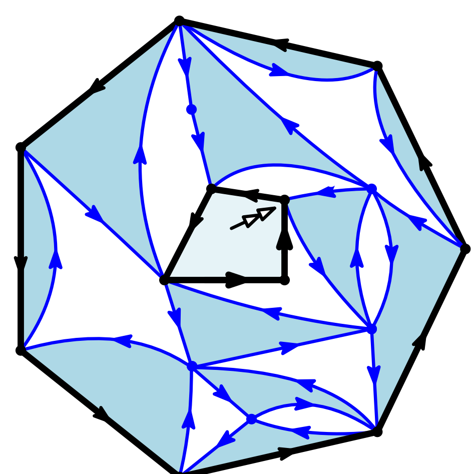


one monochromatic boundary



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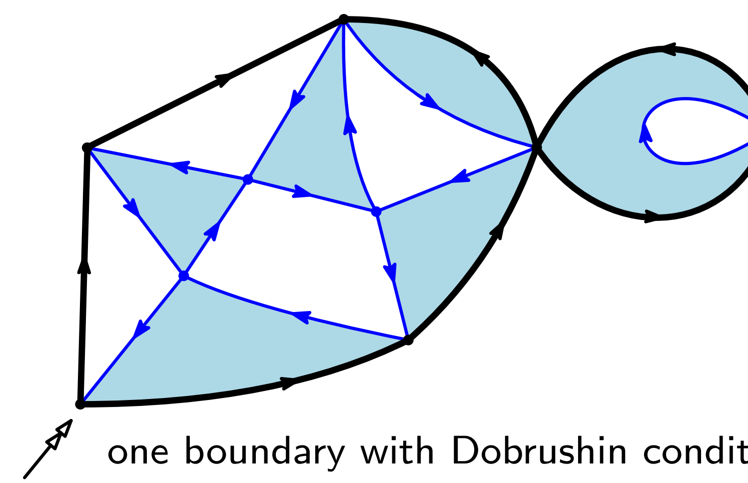

one boundary with Dobrushin condition

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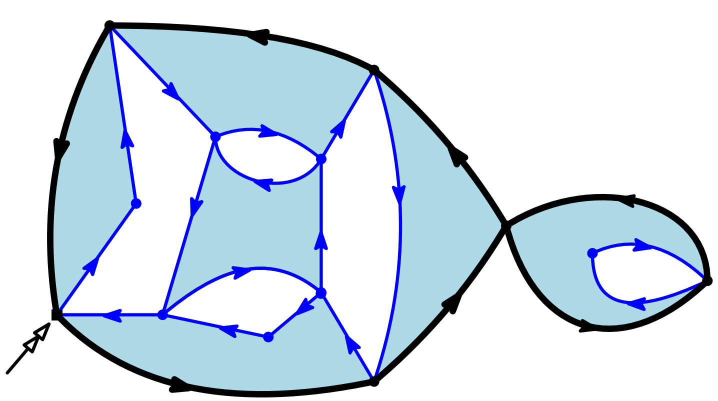

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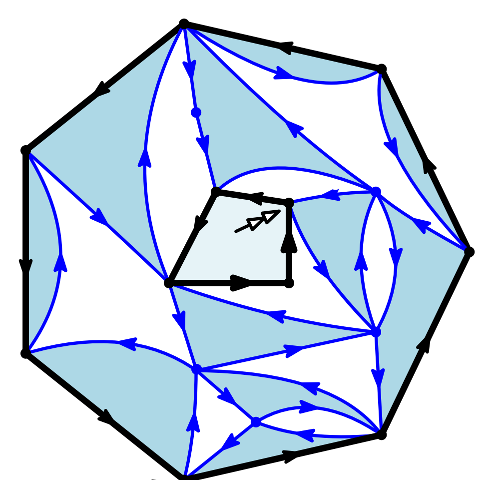

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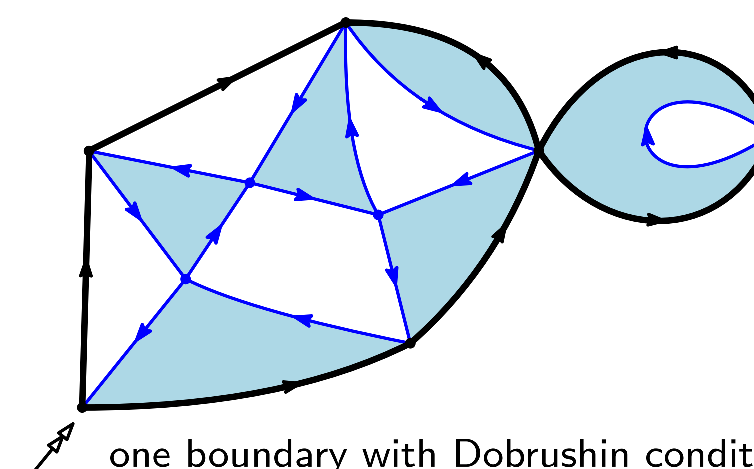

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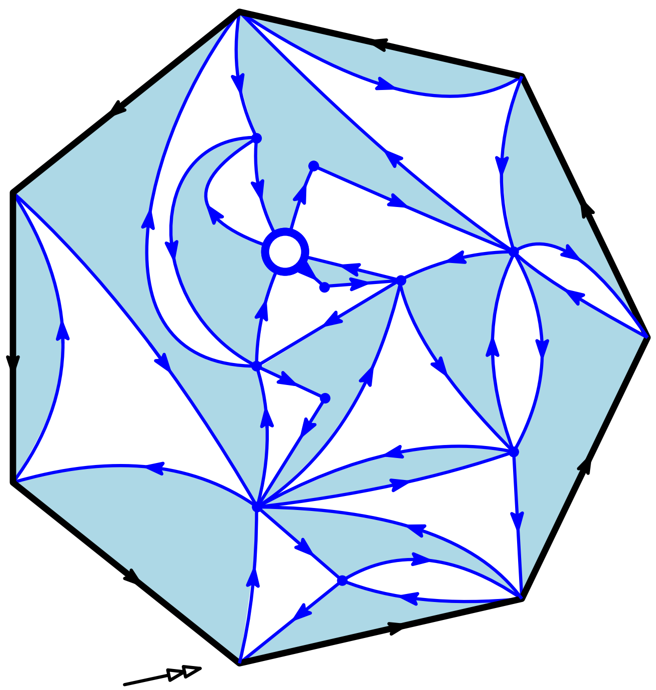
How ? By extending to hypermaps the method of **slice decomposition** introduced in [Bouttier, Guitter 2010].

First things first: Pointed disks



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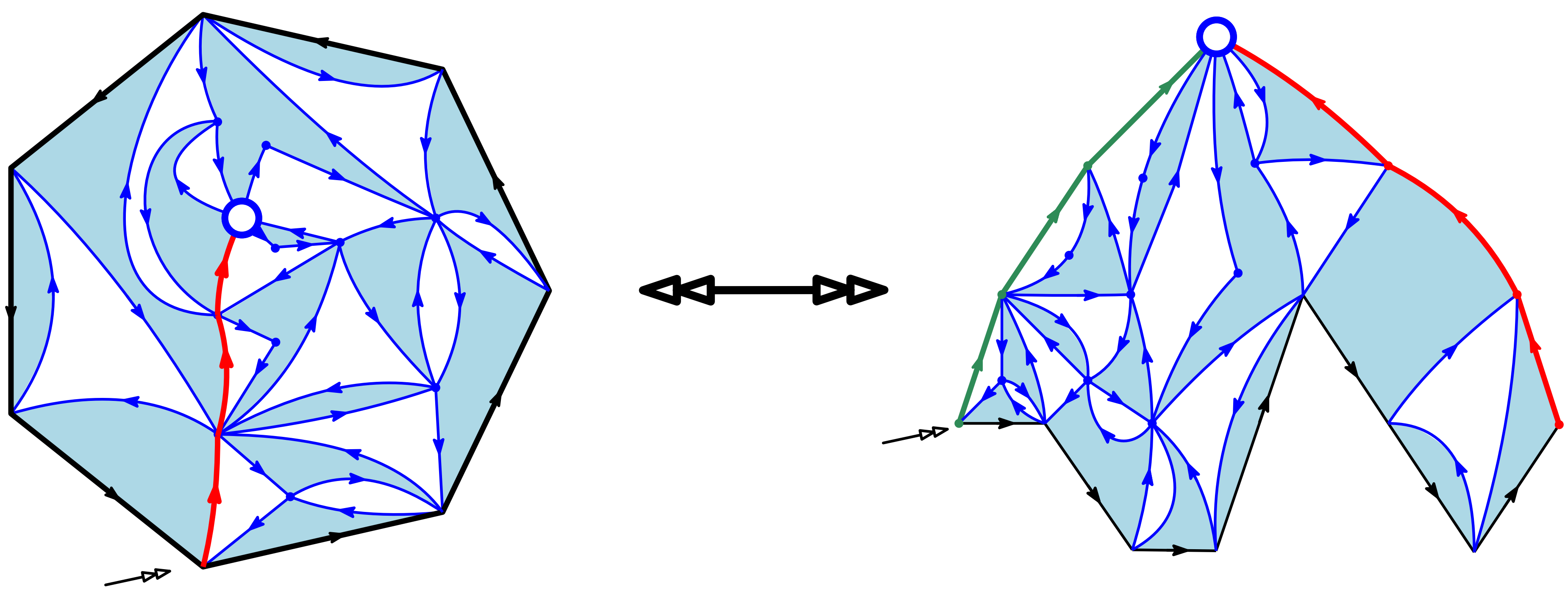

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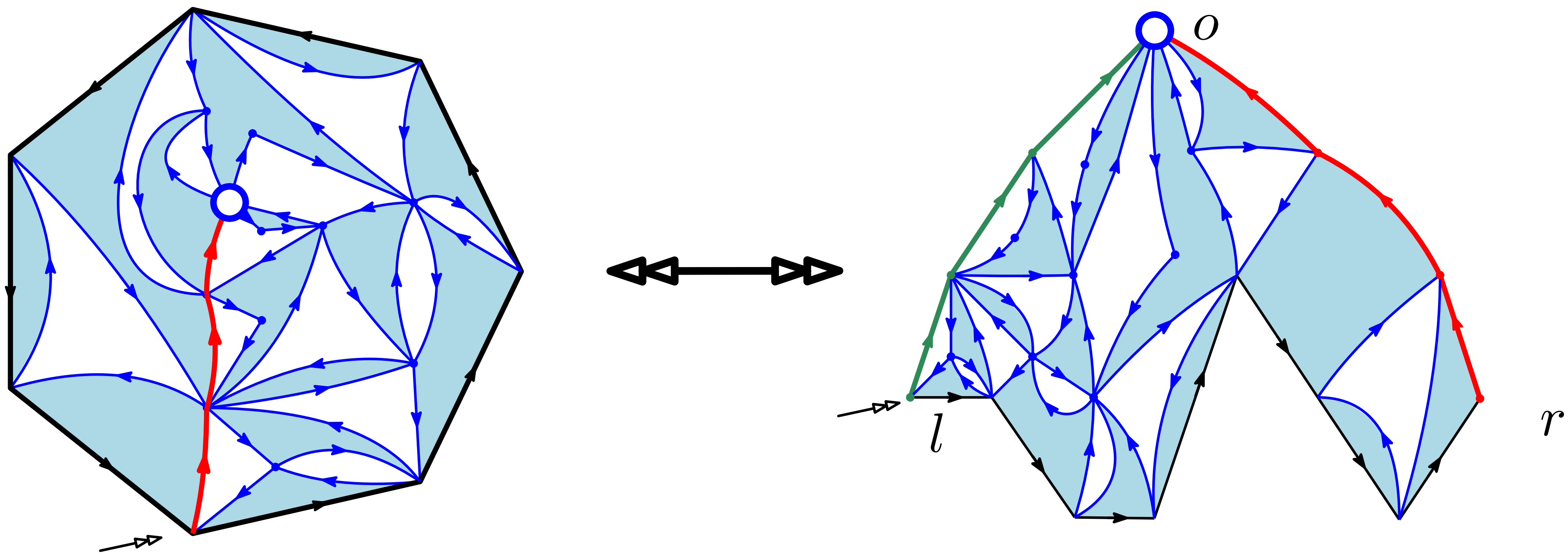

A **(hyper)-slice** is an hypermap with a boundary and 3 marked corners l, r and o such that:

- the **left boundary** from l to o is a **geodesic** (green edges)
- the **right boundary** from r to o is the **unique geodesic** (red edges),
- the **base** (between l and r) is either oriented from l to r (="type A") or from r to l (="type B") (black edges),
- and the left and the right boundaries intersect only at o.

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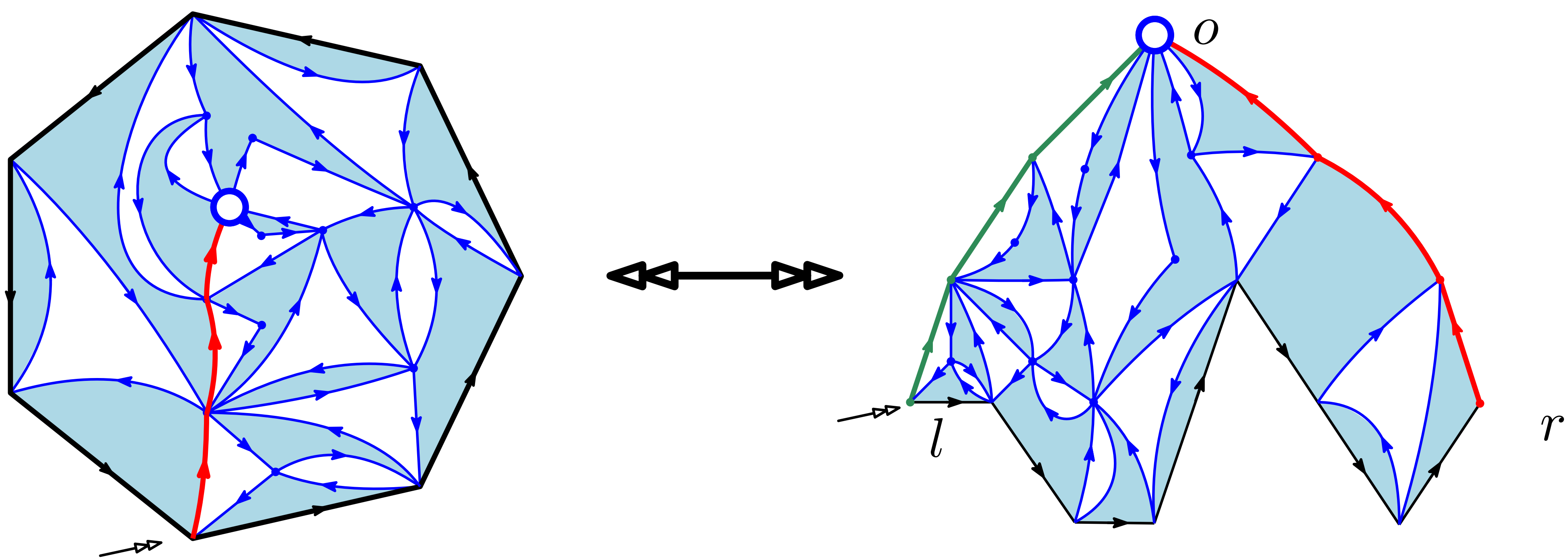

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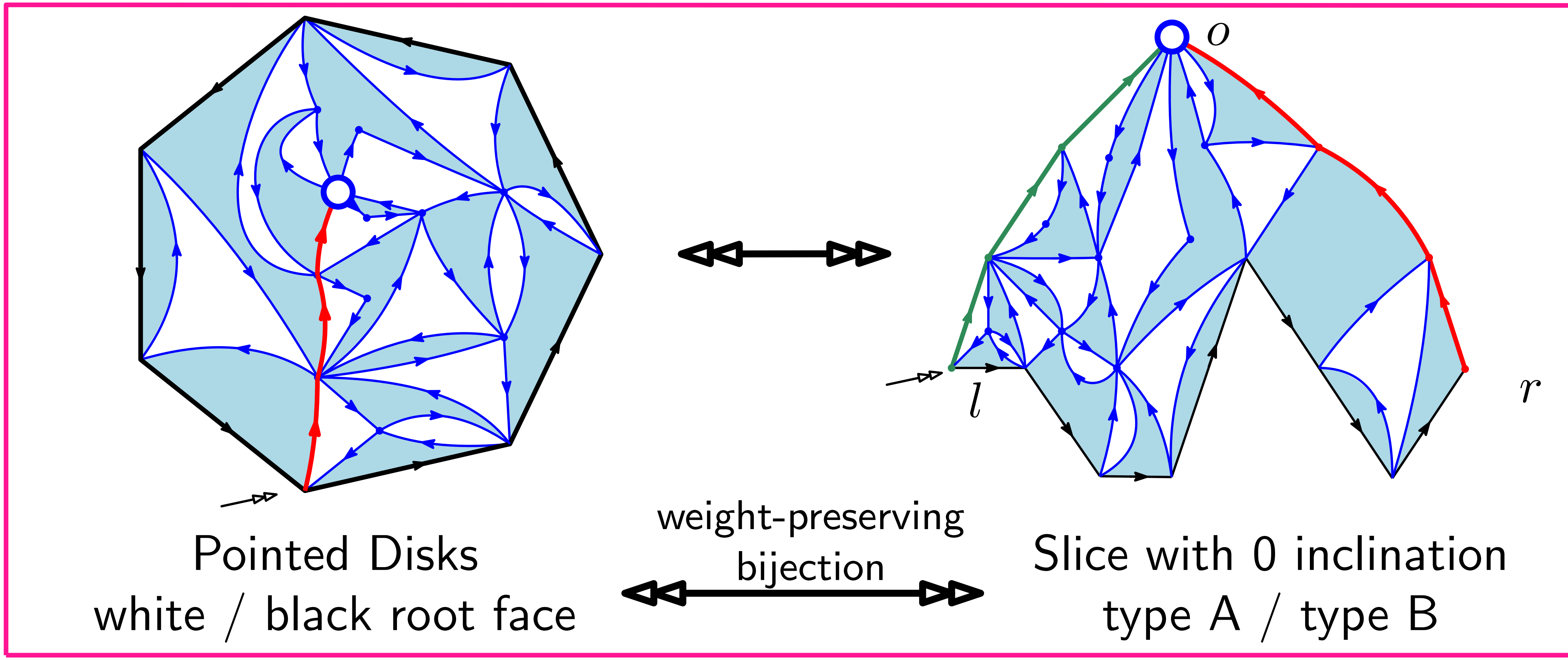

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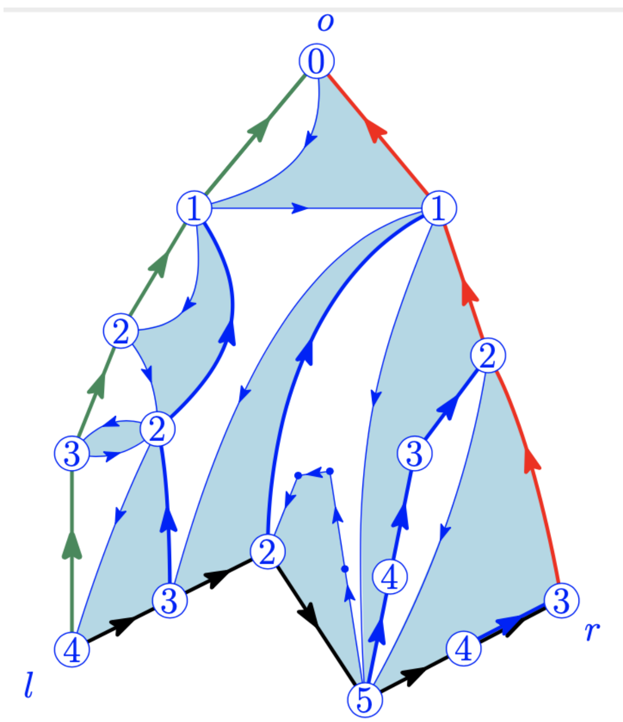
Why does this help ? Decomposition of slices

Slices can be further decomposed into “elementary slices”:



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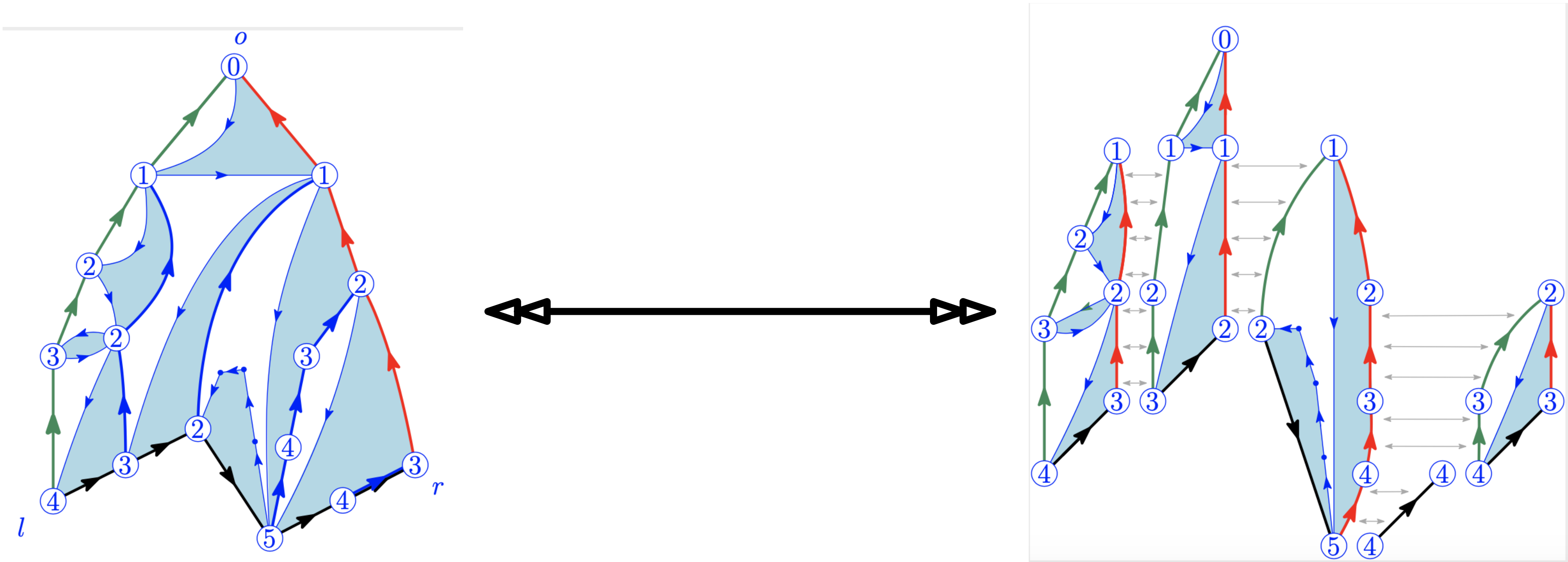

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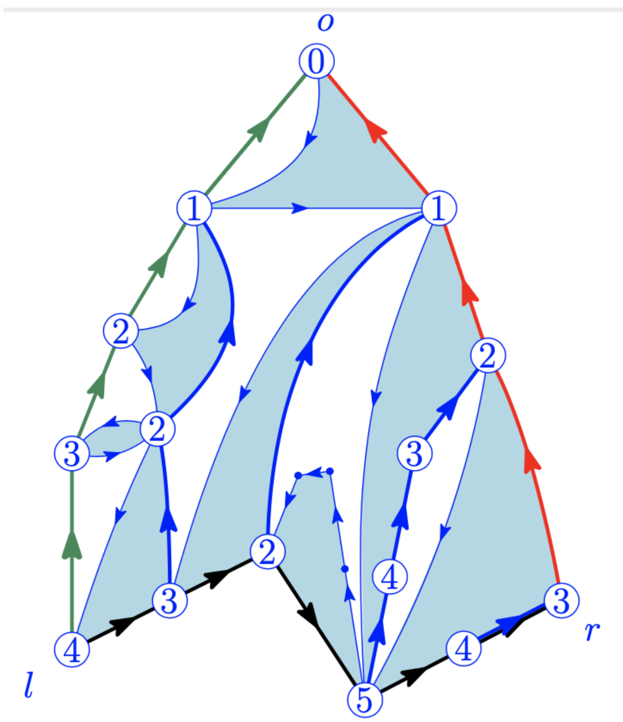

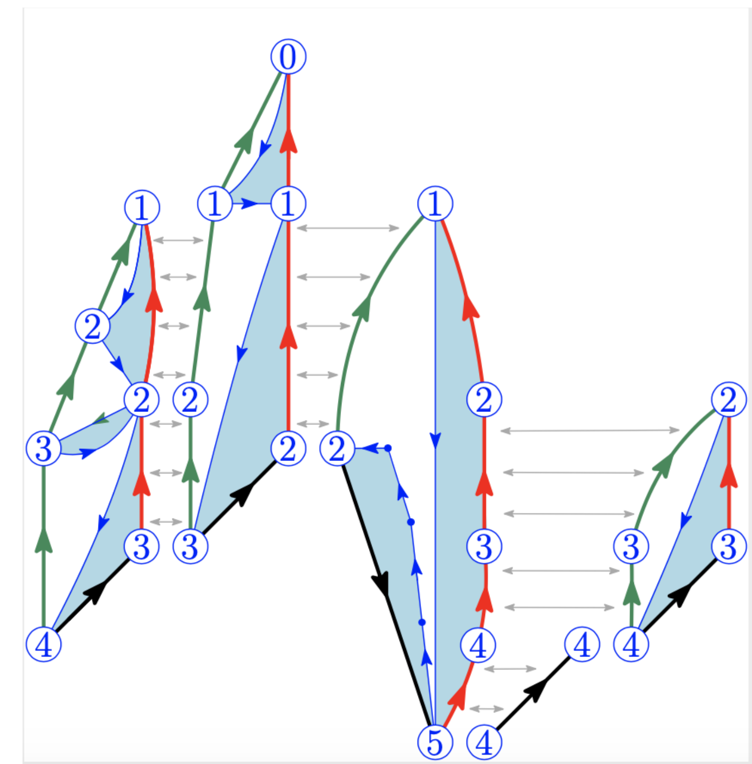

weight-preserving bijection

Type A / B slice with base of length p and inclination k

$p-$tuple of type A/B **elementary** slices s.t. sum of inclinations = k

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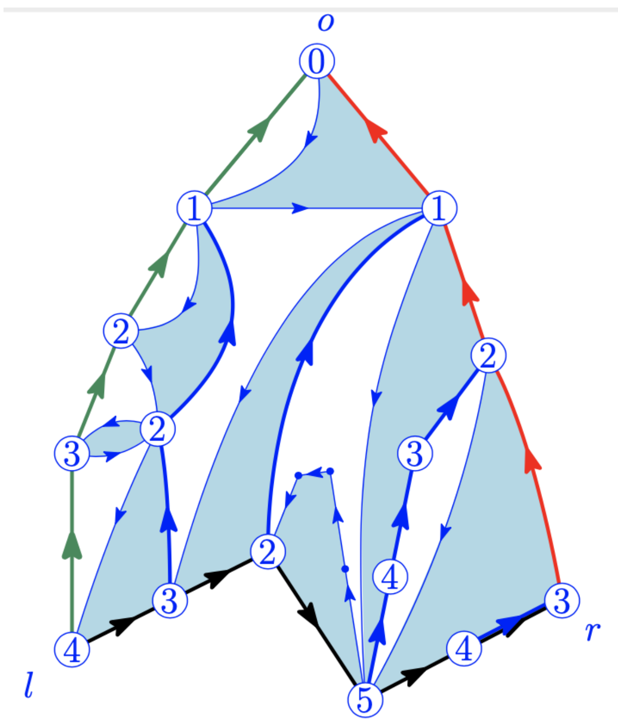

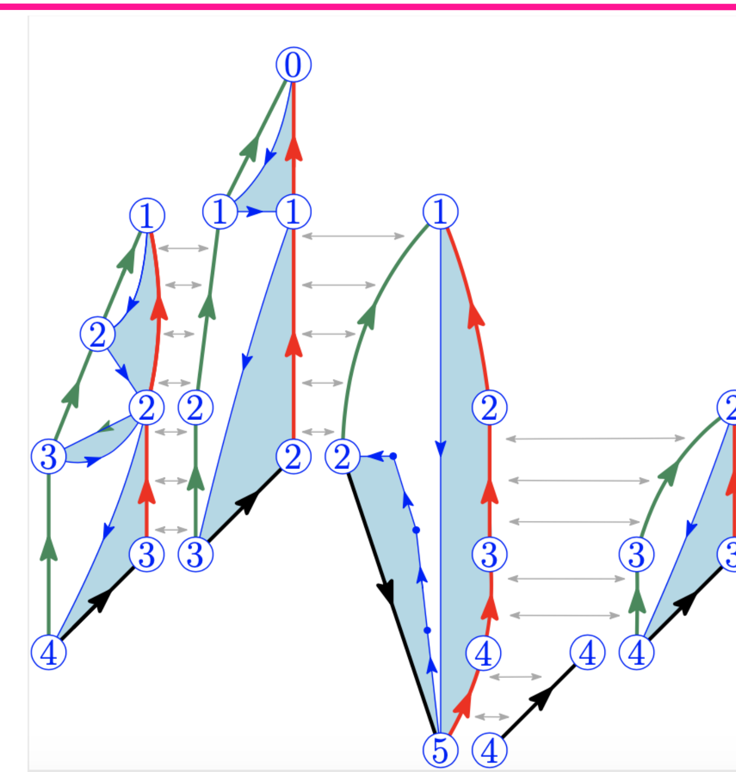

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⟺ weight-preserving bijection ⟺

$p-$tuple of type A/B **elementary** slices s.t. sum of inclinations = k

weight of a slice: $\bar{w}(\mathrm{s}) := t^{|\text{vertices of } \mathrm{s} \text{ not incident to the right boundary}|} \prod_{f \in F^{\circ}_{\text{inn}}} t^{\circ}_{\deg(f)} \prod_{f \in F^{\bullet}_{\text{inn}}} t^{\bullet}_{\deg(f)}$

Elementary slice: slice with a base of length 1.

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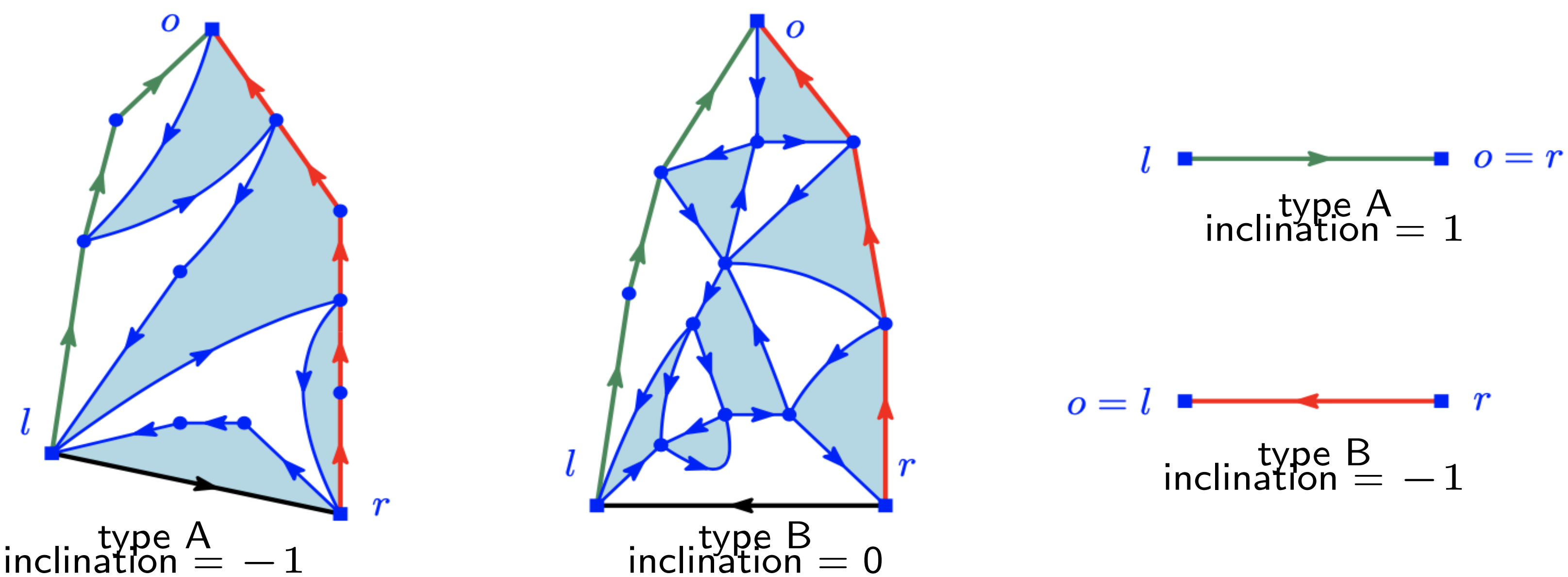
Generating series of elementary slices

For $k \in \mathbb{Z}$, $a_k, b_k :=$ generating series of elementary slices of type A/B and inclination k.



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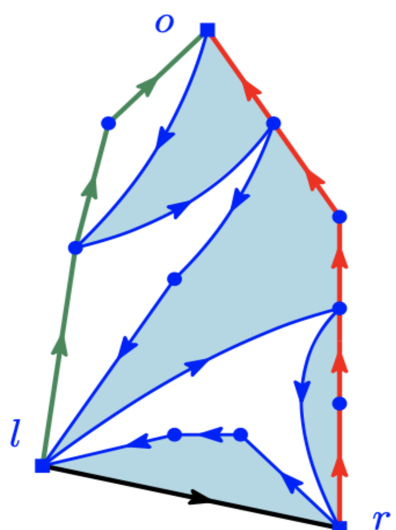
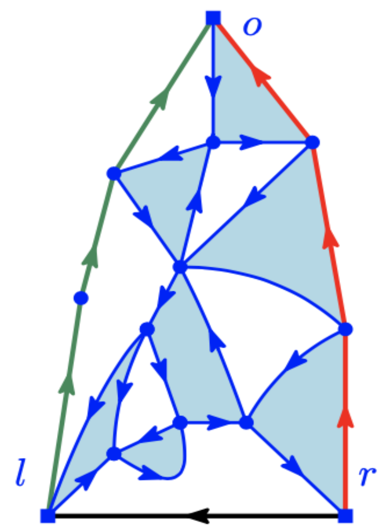


type A
inclination $= -1$



type B
inclination $= 0$

type A
inclination $= 1$

type B
inclination $= -1$

First properties :

- $a_k = b_{-k} = 0$ for $k > 1$.
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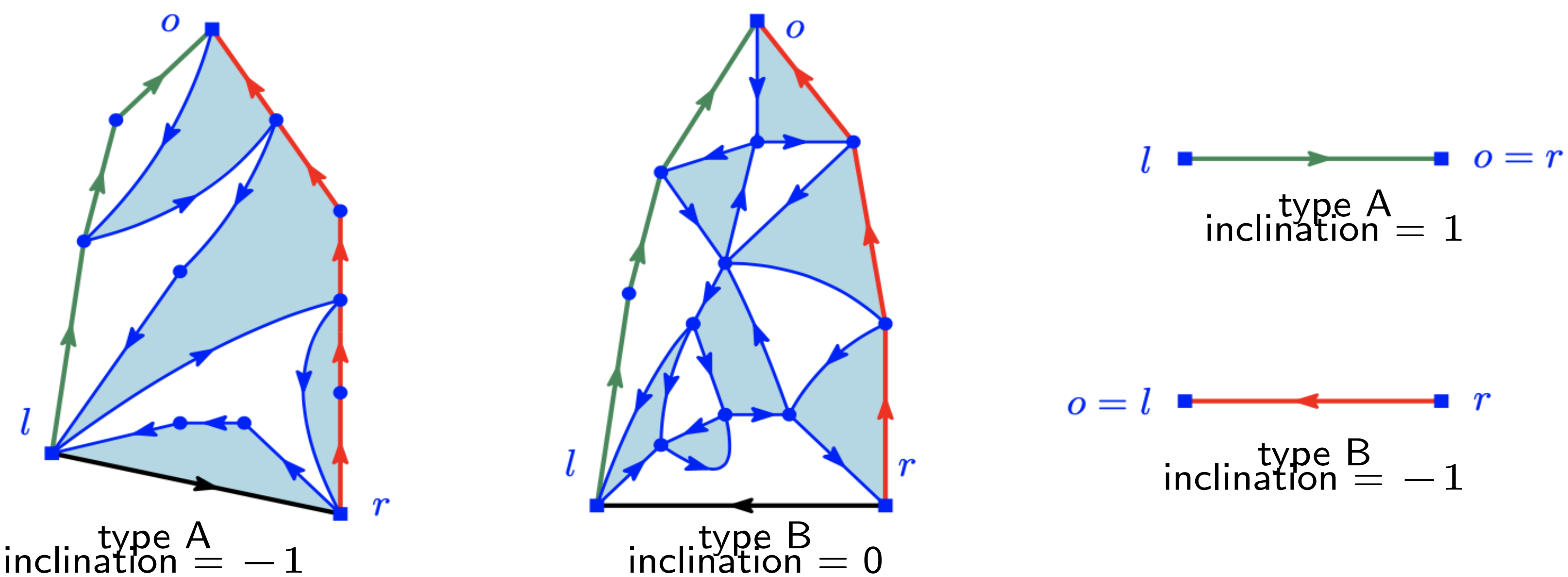

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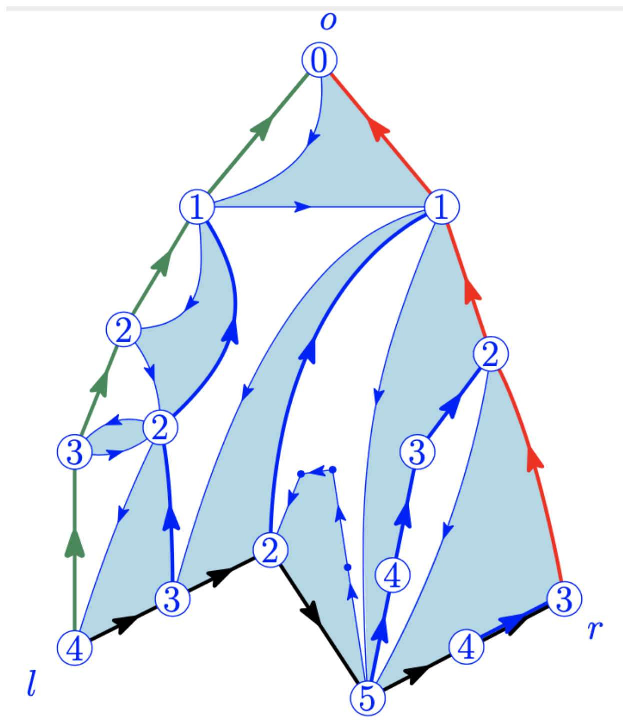
We combine all these quantities into two Laurent series:

$$x(z) := \sum_{k \leq 1} a_k z^k, \qquad y(z) := \sum_{k \geq -1} b_k z^k.$$

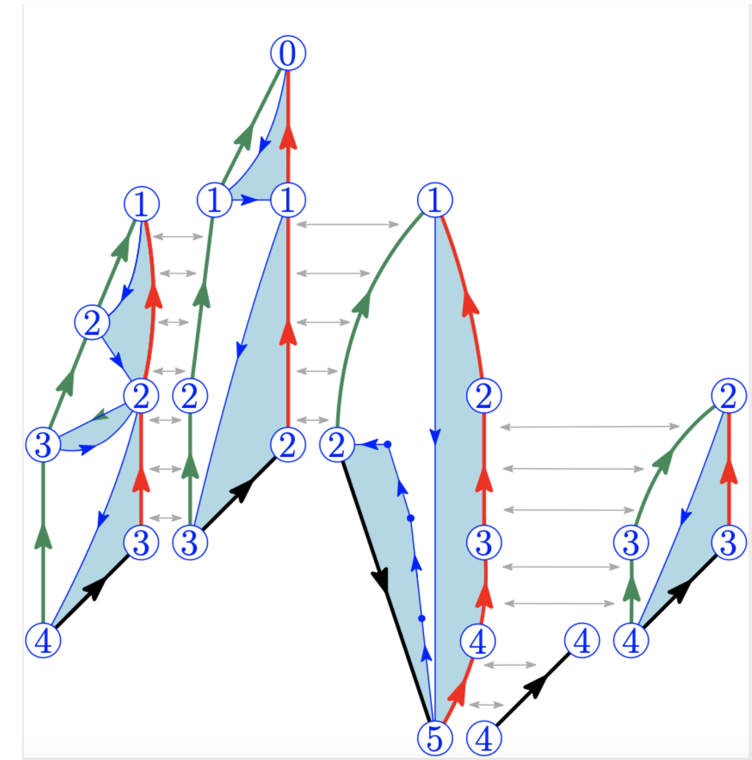
Main result:
All generating series of discussed hypermaps can be expressed in terms of $x(z)$ and $y(z)$
$=$ "spectral curve".

Generating series of slices



Type A / B slice with
base of length p and inclination k

weight-preserving
bijection

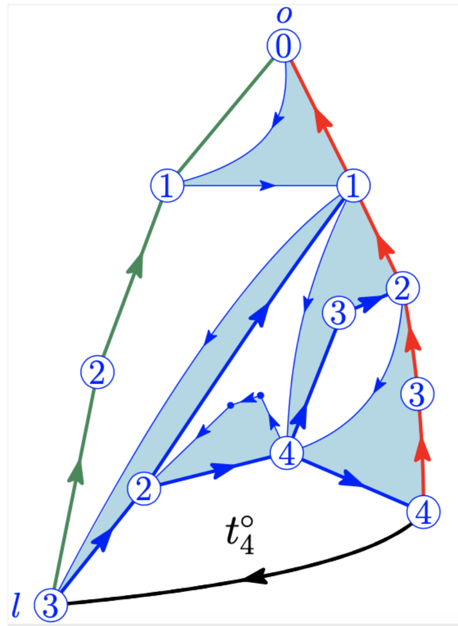


$p-$tuple of type A/B **elementary** slices
s.t. sum of inclinations = k

The generating series of slices with base of length p and inclination k is given by:

$[z^k]x(z)^p$ for type A, and $[z^k]y(z)^p$ for type B.

Why does this help ?? Decomposition of elementary slices



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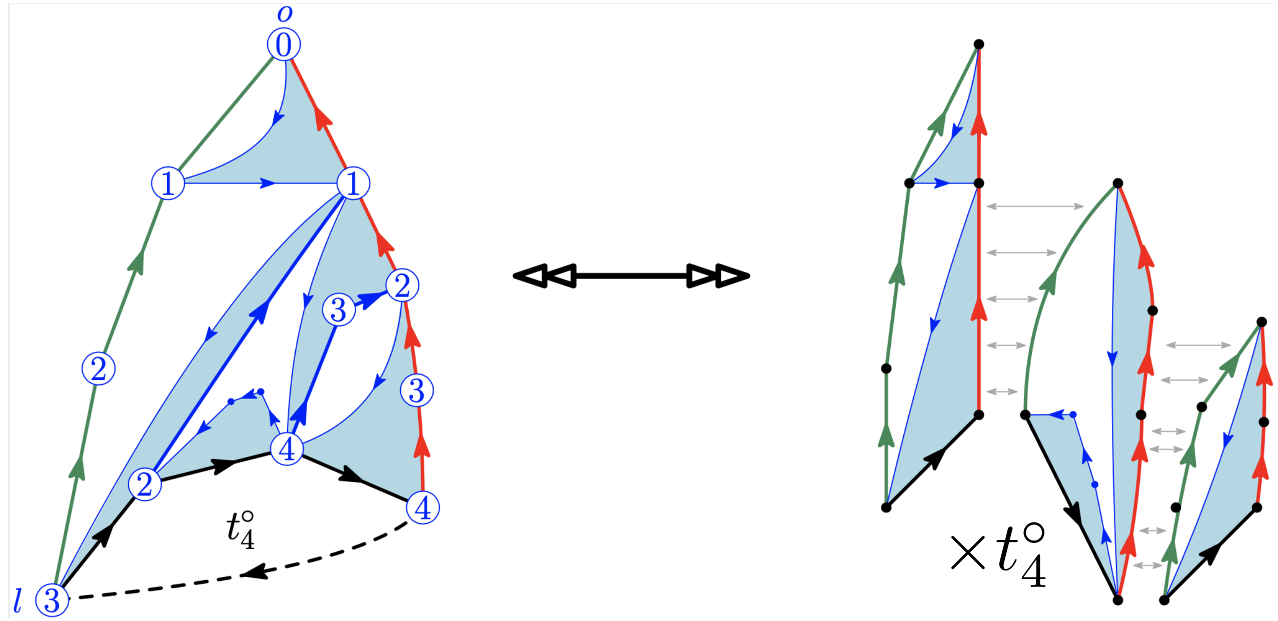

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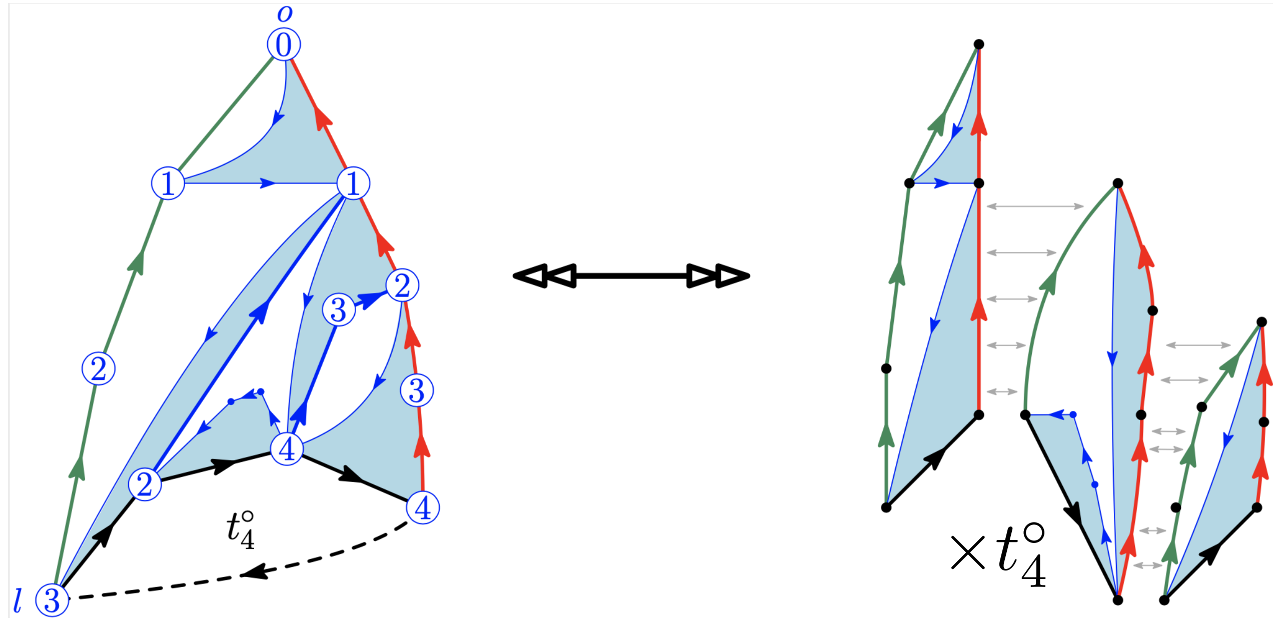

The generating series of elementary slices are uniquely determined by the following recursive system of equations:

$$a_k = t\delta_{k,1} + \sum_{d\geq 1} t_d^\bullet [z^k] y(z)^{d-1} \quad \text{for } k \leq 1$$

$$b_{-1} = 1 \ \text{ and } \ b_k = \sum_{d\geq 1} t_d^\circ [z^k] x(z)^{d-1} \quad \text{for } k \geq 0$$

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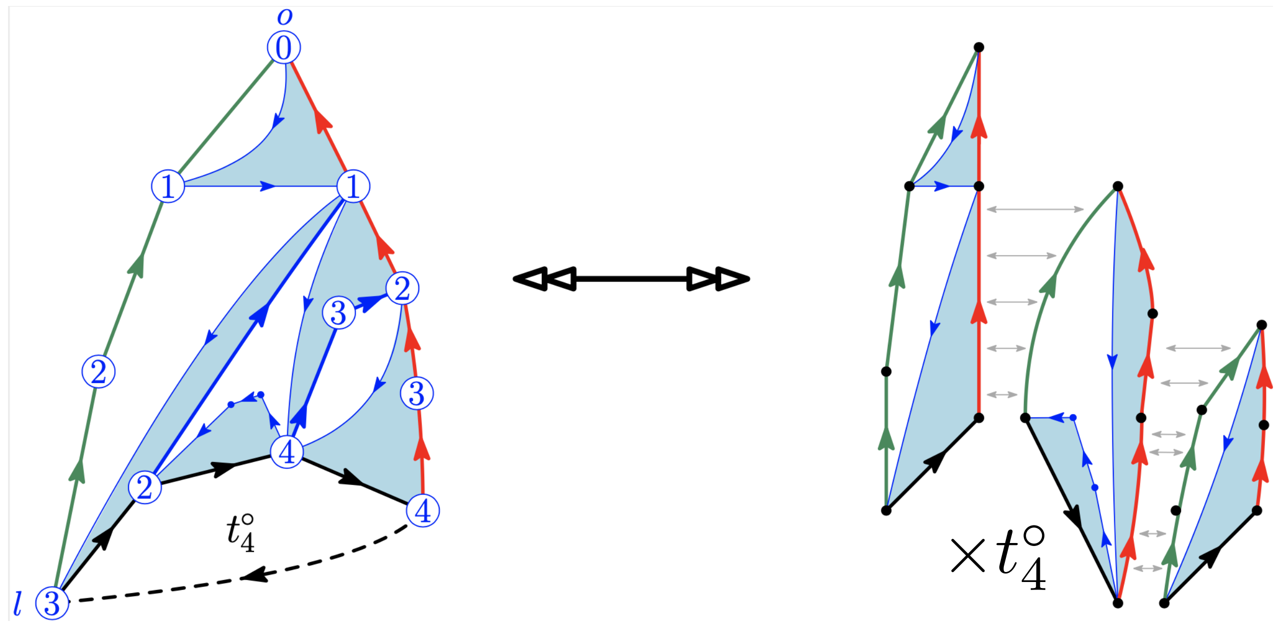

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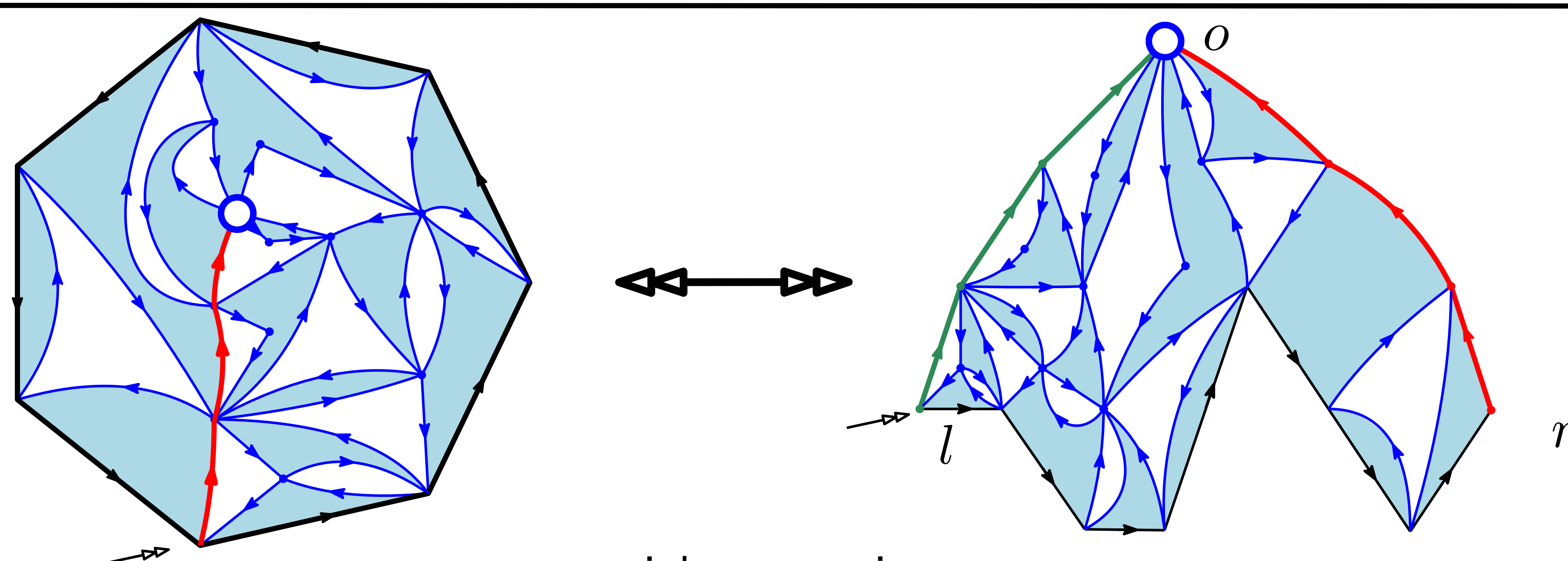
→ This system is algebraic when the degree of the faces are assumed to be bounded (i.e. $t_k^\circ = t_k^\bullet = 0$ for large k).

→ Same system of equations as [Bousquet-Mélou, Schaeffer 02] + the system of [Bouttier, Di Francesco, Guitter 04] can be recovered using an additional combinatorial construction.

Coming back to pointed disks



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Pointed Disks
white / black root face

Slice with 0 inclination
type A / type B

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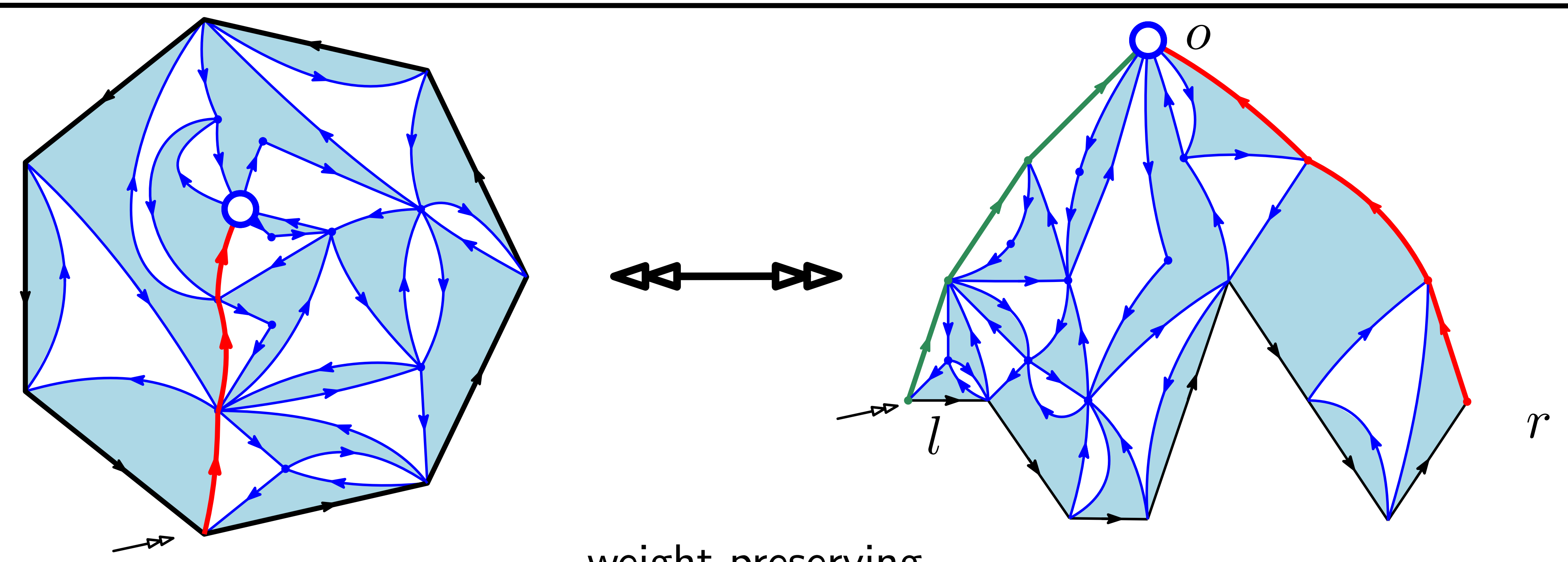

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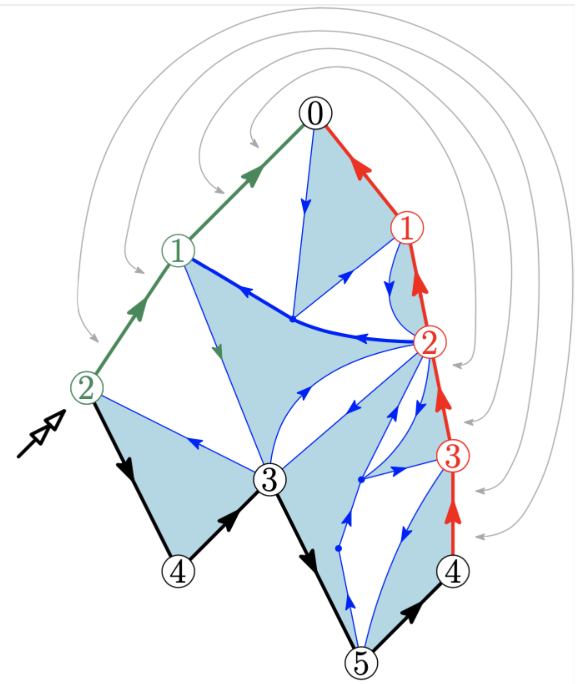
$F_p^\circ, F_p^\bullet$:= generating series of hypermaps with a monochromatic white (resp. black) boundary of degree p.

We have:

$$\frac{d}{dt}F_p^\circ = [z^0]x(z)^p, \qquad \text{resp. } \frac{d}{dt}F_p^\bullet = [z^0]y(z)^p.$$

Two boundaries: trumpets and slices with increment $\neq 0$



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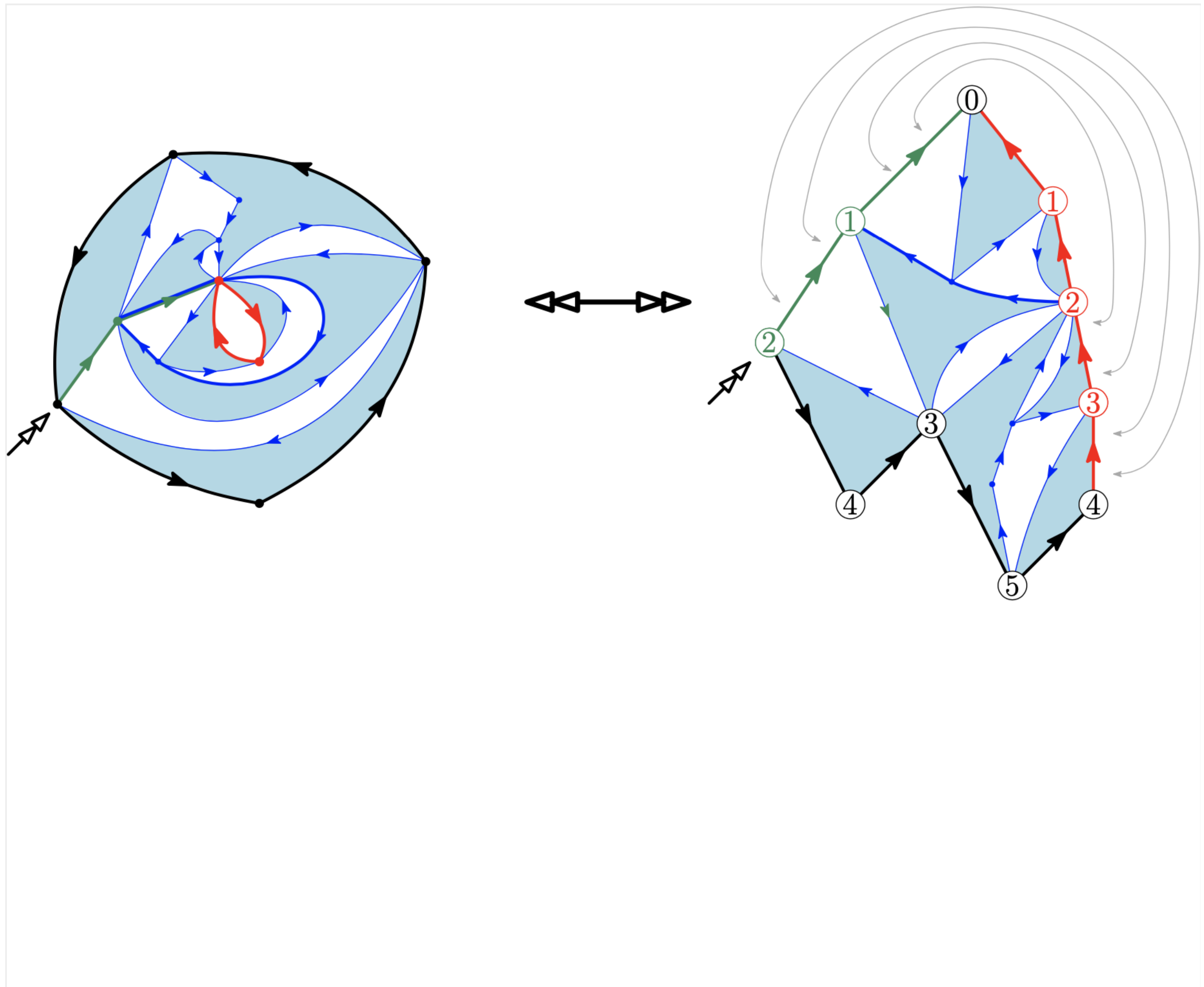

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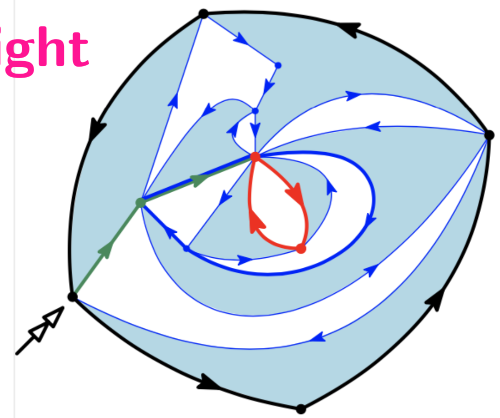
Trumpet : Hypermap with 2 monochromatic boundaries: one rooted and one **strictly tight**

:= The boundary of the tight face is the unique shortest separating cycle between both boundaries.



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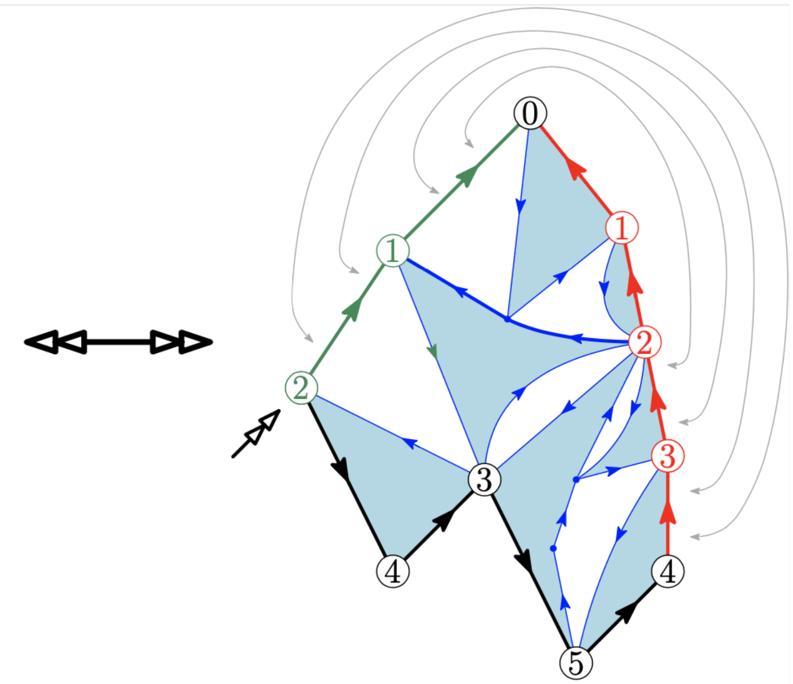

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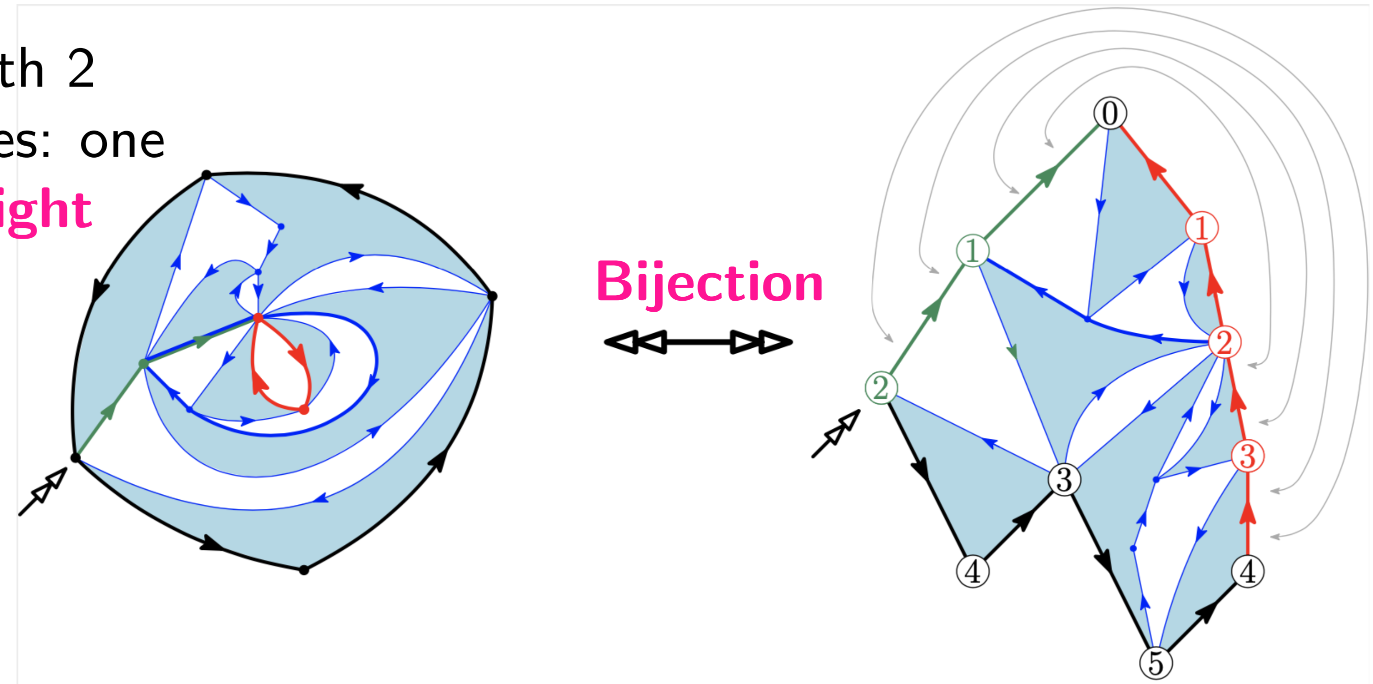

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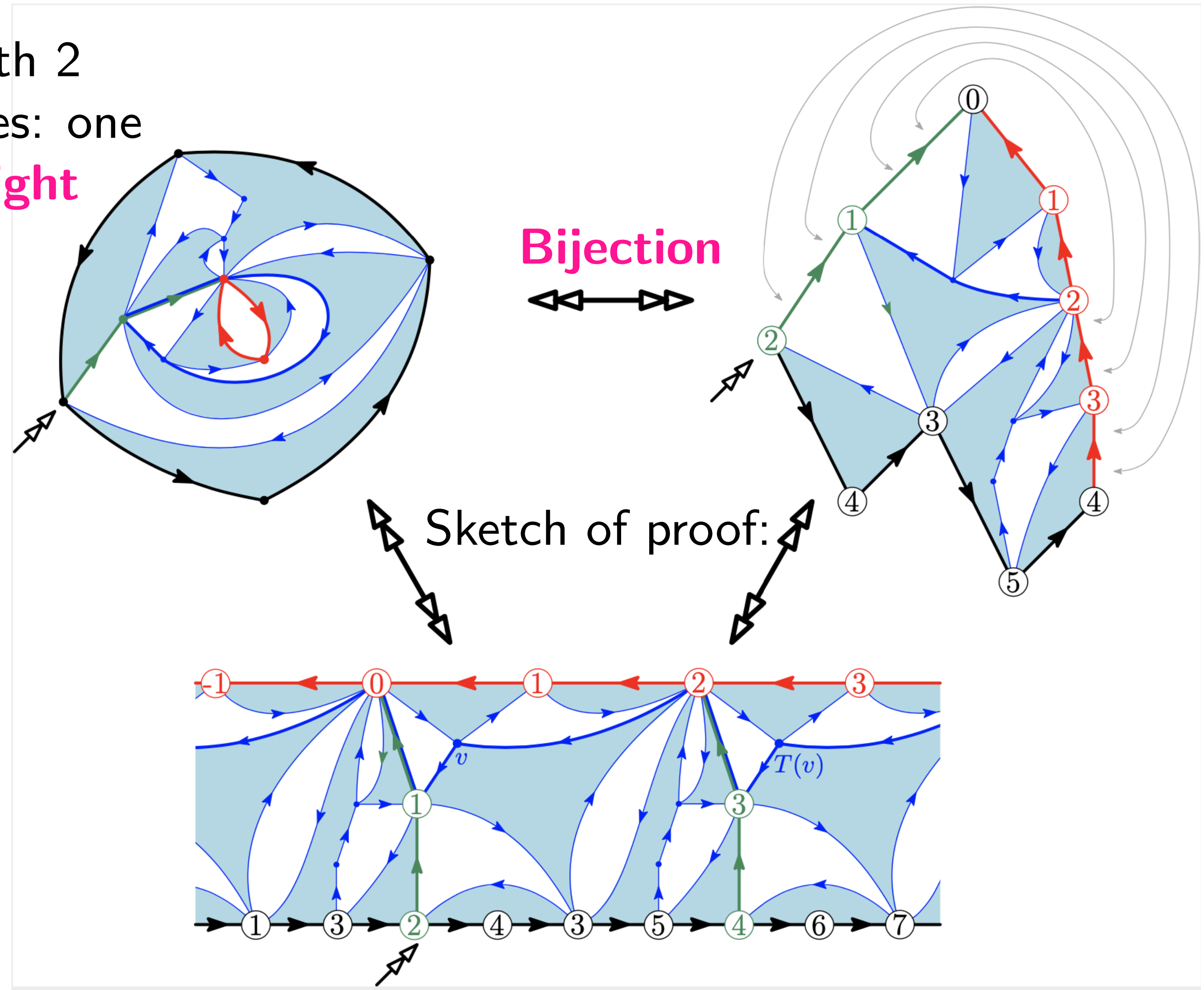

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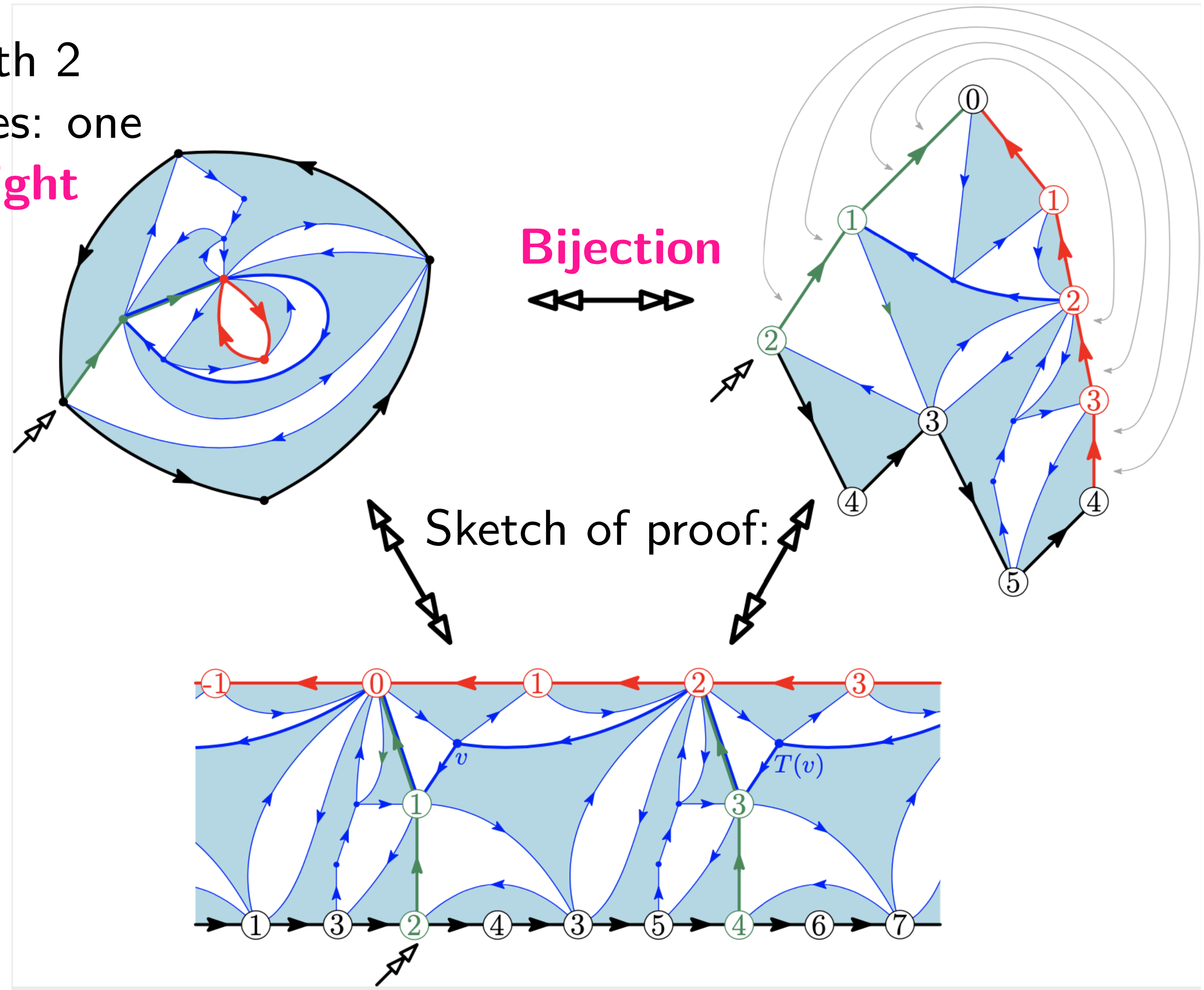

Slice with increment < 0.

Remark: Similar result for slices with increment > 0 and trumpets with a **tight face**.

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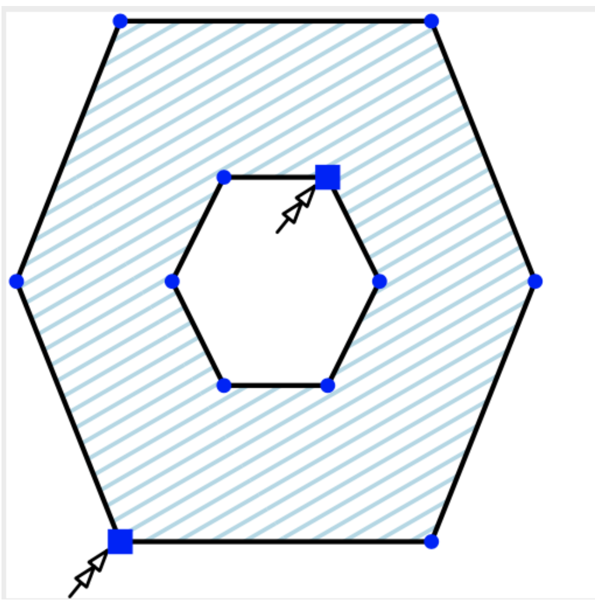
Two monochromatic boundaries: general case

An hypermap with two monochromatic boundaries can be decomposed along the "most-inside" shortest separating cycle: we get a pair of trumpets (one strict and the other not).



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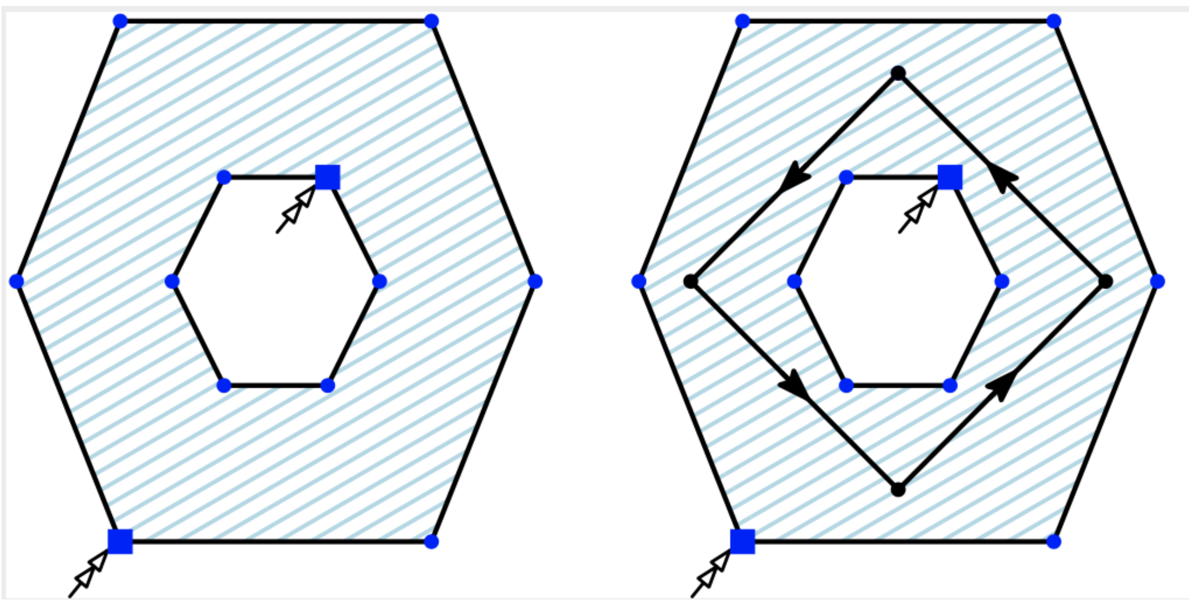

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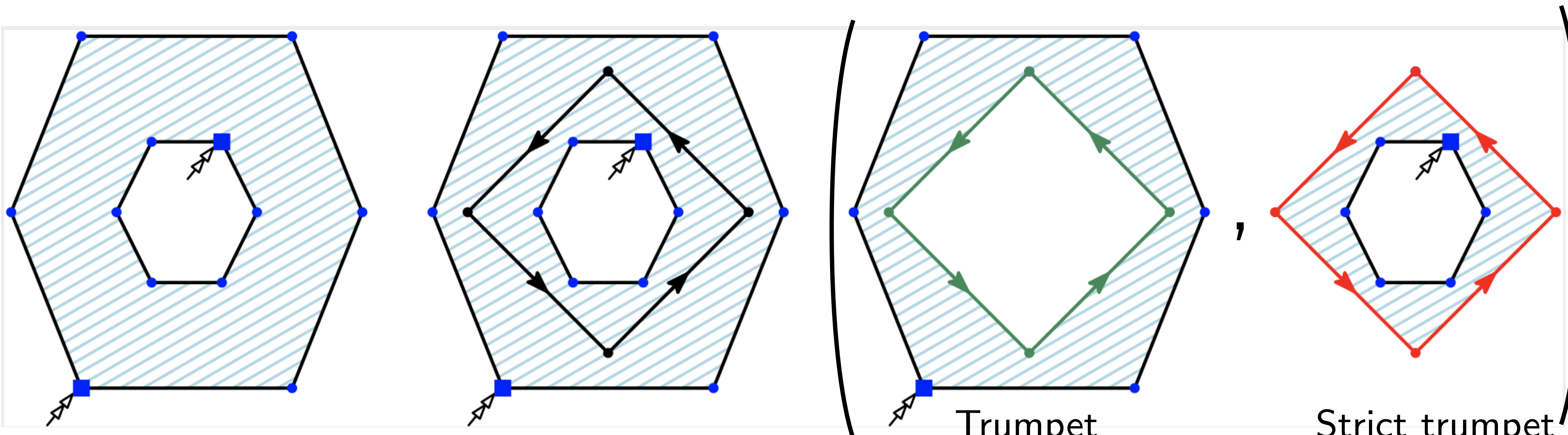

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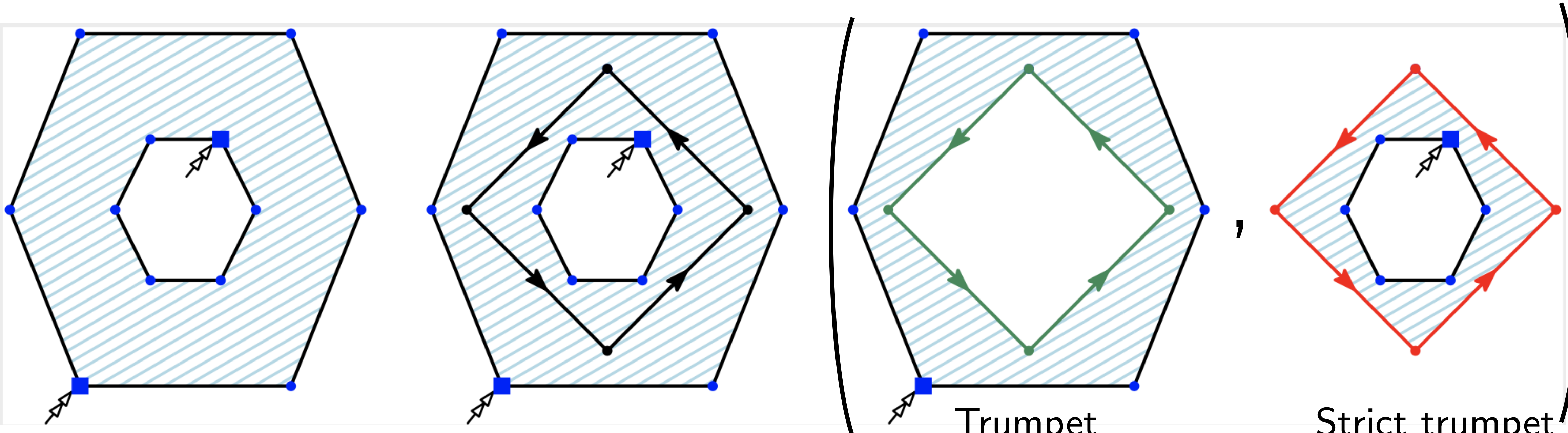

The generating series of hypermaps with two monochromatic boundaries are given by:

$$F^{\circ\circ}_{p,q} = \sum_{h\geq 1} h\left([z^h]x(z)^p\right)\left([z^{-h}]x(z)^q\right), \qquad F^{\circ\bullet}_{p,q} = \sum_{h\geq 1} h\left([z^h]x(z)^p\right)\left([z^{-h}]y(z)^q\right),$$

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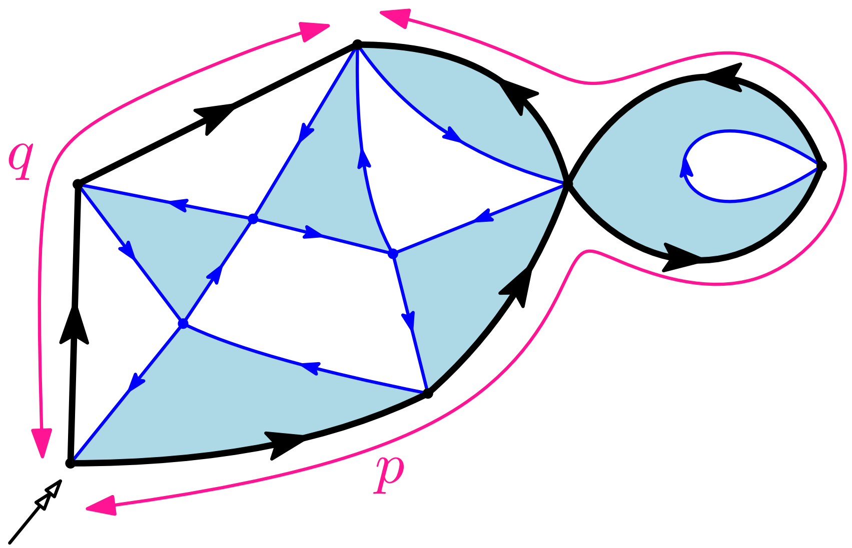
One more result and a conclusion:

Generating series of hypermaps with a **Dobrushin boundary condition**:



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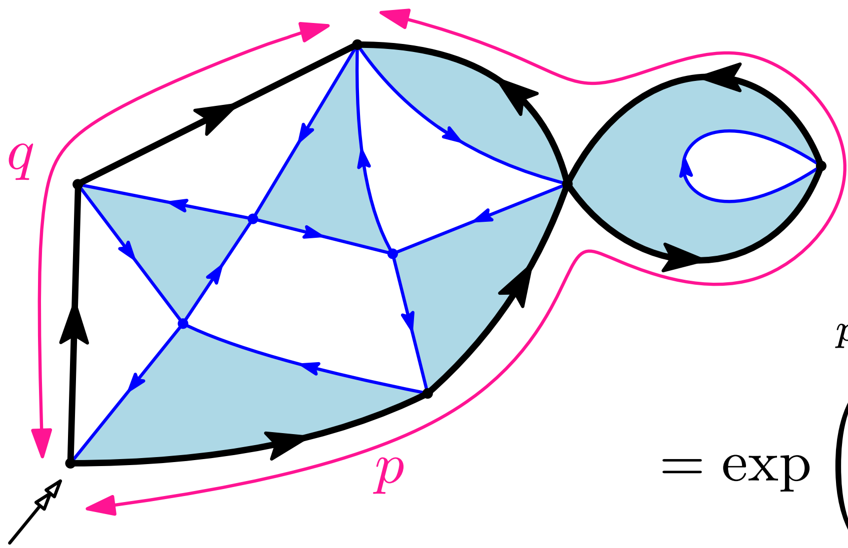

$$\sum_{p,q\geq 0} \frac{F^{◑}_{p,q}}{x^{p+1}y^{q+1}}$$

$$= \exp\left(\sum_{h\in\mathbb{Z}} h\left([z^h]\ln\left(1-\frac{x(z)}{x}\right)\right)\left([z^{-h}]\ln\left(1-\frac{y(z)}{y}\right)\right)\right) - 1$$

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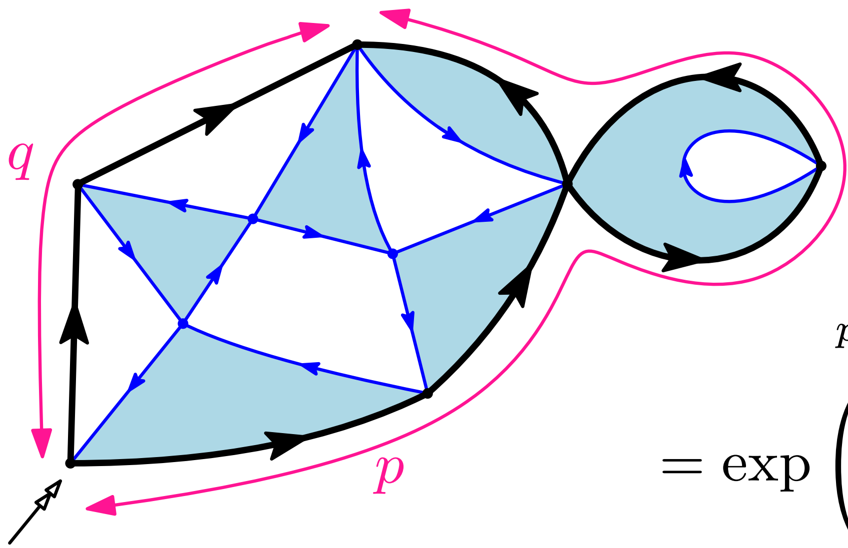

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We gave **bijective derivation** of enumerative formulas for hypermaps with one or two boundaries.

But even more mysterious formulas are available – for hypermaps with more boundaries or with any boundary conditions – which still lack a bijective derivation.

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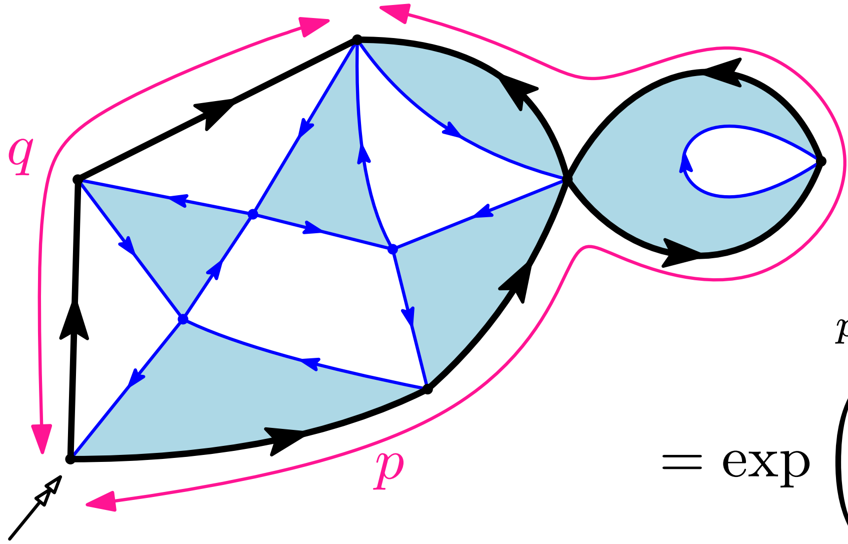

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To be followed...

THANK YOU !