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Three facets of Gravity,
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Effective BV quantisation of Gravity with and without boundary

Exercise Sheet 11.05.2023

The goal of the first two exercises is to construct the BV and BFV action for a simple example, one-dimensional gravity (a theory classically equivalent to classical mechanics [Bonus exercise: Can you show this as well?]).

Exercise 1. The action of one-dimensional gravity is

$$S[q,g] := \int_a^b \left(\frac{1}{\sqrt{g}} T(\dot{q}) - \sqrt{g} V(q) + \sqrt{g} E \right) \mathrm{d}t, \tag{1}$$

where the fields are $q\colon [a,b] \to U$ and $g\colon [a,b] \to \mathbb{R}_{>0}$ for $U \subset \mathbb{R}^n$ an open subset.

1. Compute the variation of the action (1).
2. From the variation of the action extract the Euler–Lagrange equations and verify that the resulting 1-form is
$$\check{\alpha} = \sum_i \frac{m\dot{q}^i}{\sqrt{g}} \mathrm{d}q^i.$$
3. Introducing $p_i := m\dot{q}^i/\sqrt{g}$, verify that the space of boundary fields F^∂ is given by (two copies of) T^*U with canonical symplectic form.
4. Rewrite the EL equations in terms of the new variables and identify the evolution equations and the constraints. Then write the reduced phase space as a quotient.

Exercise 2. Let $\xi \in \Gamma[1](TU)$, i.e. a shifted vector field parametrizing reparametrization of the theory (i.e $1d$ diffeomorphisms). Define the BV operator Q as follows:

$$Qq = \xi\dot{q} \qquad Qg = \xi\dot{g} + 2g\dot{\xi} \qquad Q\xi = \xi\dot{\xi}.$$

1. Verify that $Q^2 = \frac{1}{2}[Q,Q] = 0$ and that $QS = 0$ up to boundary terms.

Let now

$$\omega_{\mathrm{BV}} = \int_a^b \left(\sum_i \delta q_i^+ \delta q^i + \delta g^+ \delta g + \delta\xi^+ \delta\xi \right) \mathrm{d}t.$$

be the BV symplectic form, where q^+, g^+ and ξ^+ are the antifields of q, g and ξ respectively and let the BV action be

$$\mathcal{S}[q,q^+,g,g^+,\xi,\xi^+] = S[q,g] + \int_a^b \left(\sum_i q_i^+ \xi \dot{q}^i + g^+(\xi\dot{g} + 2g\dot{\xi}) - \xi^+\xi\dot{\xi} \right) \mathrm{d}t.$$

2. Find Qq^+ Qg^+ and $Q\xi^+$ such that $\iota_Q\delta\omega_{\mathrm{BV}} - \delta\mathcal{S}$ is a boundary term. Verify that the resulting boundary term is

$$\check{\alpha} = \left(\frac{m\dot{q}}{\sqrt{g}} + q^+\xi\right)\cdot \mathrm{d}q + g^+\xi\mathrm{d}g + (\xi^+\xi - 2g^+g)\mathrm{d}\xi.$$

3. Compute $\check{\omega} = \delta\check{\alpha}$ and find its kernel.

It is possible to see that the reduced space of boundary fields $\mathcal{F}^\partial$ can be identified with $T^*(\mathbb{R}^n\times\mathbb{R}[1])$ with base coordinates q, c and fiber coordinates p, b and canonical 1-form $\alpha^\partial = p\cdot \mathrm{d}q + b\,\mathrm{d}c$. The projection map is defined by

$$\begin{aligned} p &= \frac{m\dot{q}}{\sqrt{g}} + q^+\xi,\\ b &= \frac{1}{\sqrt{g}}(\xi^+\xi - 2g^+g),\\ c &= \sqrt{g}\,\xi. \end{aligned} \tag{2}$$

Let now $E = \xi\frac{\partial}{\partial\xi} - 2\xi^+\frac{\partial}{\partial\xi^+} - g^+\frac{\partial}{\partial g^+} - \sum_i q_i^+\frac{\partial}{\partial q_i^+}$.

4. Using the results of point 2, compute $\check{S} = \iota_Q\iota_E\check{\omega}$.

5. Deduce that $\check{S}$ is the pullback along the projection (2) of

$$S^\partial = \left(\frac{||p||^2}{2m} + V(q) - E\right)c.$$

Exercise 3. In this exercise we show how it is possible to define the BV-Laplacian using odd Fourier transforms. Let M be an n-dimensional manifold and fix a volume form

$$\mathrm{Vol} = \rho dx^1\wedge\cdots\wedge dx^n.$$

Let also $T[1]M$ be its graded tangent bundle, with coordinate x^μ of degree 0 on the base and ξ^μ of degree 1 on the fiber and let $D = \xi^\mu\frac{\partial}{\partial x^\mu}$ be a degree 1 vector field.

1. Observe that $D^2 = 0$.

Let now $f = f(x,\xi) \in C^\infty(T[1]M)$ and define the *odd Fourier tranform* of f as the function on $C^\infty(T^*[-1]M)$ defined by

$$F[f](x,\psi) = \int d^n\xi\rho^{-1}e^{\psi_\mu\xi^\mu}f(x,\xi).$$

where ψ^μ are coordinates of degree -1 and ρ is a fixed volume form. Define also

$$F^{-1}[\widetilde{f}](x,\xi) = (-1)^{n(n+1)/2}\int d^n\psi\rho e^{-\psi_\mu\xi^\mu}\widetilde{f}(x,\psi).$$

2. Prove that $F^{-1}[F[f]] = f$.

3. Prove that there exists an operator Δ such that

$$F[Df] = (-1)^n \Delta F[f]$$

and its explicit coordinate expression is

$$\Delta = \rho^{-1} \frac{\partial^2}{\partial \psi_\mu \partial x^\mu} \rho.$$

4. Prove that Δ is a BV-Laplacian, i.e. show that $\Delta(fg) = (\Delta f)g + (-1)^{\deg(f)} f(\Delta g) + (-1)^{\deg(f)}\{f, g\}$, where $\{\cdot,\cdot\}$ is the BV bracket, defined as

$$f, g = -(-1)^{\deg(f)} \frac{\partial}{\partial \xi^\mu} f \frac{\partial}{\partial x^\mu} g + \frac{\partial}{\partial x^\mu} f \frac{\partial}{\partial \xi^\mu} g$$

for every function $f, g \in C^\infty(T[1]M)$.